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REVIEW ARTICLE

SOME ANALYTICAL PRELIMINARIES: A STUDY

Some Analytical Preliminaries: A Study

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JACOBI POLYNOMIALS

A preliminary information which partially can be found . First of all, we recall that the classical Jacobi polynomial is the *k*-th member of the sequence of polynomials which are orthogonal on [-1; 1] with respect to the Jacobi weight.

$w_{\alpha\beta}$ (u) = $(1 – u)^{\alpha} (1 + u)^{\beta}$ (α,β >-1)

or,equivalently, to the normalized Jacobi weight

$И_{\alpha\beta}$ (u) = $w_{\alpha\beta}$ (u) / $r_{\alpha\beta} = \int w_{\alpha\beta}$ (u). du

An explicit expression for Jacobi polynomials is

$P_k^{(\alpha,\beta)}$ (u) = $1/2^k$

$\sum (\alpha + k) (\beta + k) (u – 1)^{k-v} (u + 1)^{v}$

deg $P_k^{(\alpha,\beta)} = k$, $P_k^{(\alpha,\beta)} (1) = (\alpha + k)$

$P_k^{(\alpha,\beta)} (-u) = (-1) . P_k^{(\alpha,\beta)} (u)$

In particular, the polynomials $Pk^{(\alpha,\ \alpha)}$ (u) are even for even k and odd for odd k. The latter polynomials are in essence the Gegenbauer polynomials. More precisely, the Gegenbauer (ultra spherical) polynomial is defined as

$C^v_k (u) = r (v + ½) r (2v + k) Pk^{(v – ½ , v-1/2)} (u) / r (2v) r (v + k + ½)$,v>- ½

so that

deg $C^v_k = k$, $C^v_k (1) = (2v + k – 1 /k)$

In addition ,

$C^v_k (-1) = (-1)^k (2v + k – 1 /k)$

With a fixed v the Gegenbauer polynomials (Cv k (u)) k=0 are orthogonal on [-1; 1] with respect to the weight

$w_{v-1/2\ ,\ v-1/2} (u) = (1 – u^2)^{v-1/2}$

We especially need in the Gegenbauer polynomials with v = *q*-2/ 2 *; q*∈ N*; q*>= 1*:* They are orthogonal with respect to the weight

$$w_q(u) = w_{q-3/2\ ,\ q-3/2}\ (u) = (1 - u^2)^{q-3/2}$$

or , equivalently to,

$$\Omega_q\ (u) = w_q\ (u)\ /\ r_q$$

where,

$$r_q = \int_{-1} w_q(u)\ du = r(1/2)\ r(m-1\ /2\ )\ /\ r(m/2)$$

The Cristoffel-Darboux kernel which relates to the Jacobi polynomials is

$K_t^{(\alpha,\beta)} (u,v) = \sum P_k^{(\alpha,\beta)} (u) P_k^{(\alpha,\beta)} (v)\ /\ || P_k^{(\alpha,\beta)}\ ||^2_{w\ \alpha,\beta}$

According to the Cristoffel -Darboux Formula

$K_t^{(\alpha,\beta)}(u,v) = \dfrac{1}{2^{\alpha+\beta}(2t+\alpha+\beta+2)} \cdot \dfrac{\Gamma(t+2)\,\Gamma(t+\alpha+\beta+2)}{\Gamma(t+\alpha+1)\,\Gamma(t+\beta+1)}$

$X \dfrac{P_{t+1}^{(\alpha,\beta)}(u)\,P_t^{(\alpha,\beta)}(v) - P_t^{(\alpha,\beta)}(u)\,P_{t+1}^{(\alpha,\beta)}(v)}{u-v}$

An important particular case is

$K_t^{(\alpha,\beta)}(u) = \dfrac{1}{2^{\alpha+\beta+1}} \cdot \dfrac{\Gamma(t+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(t+\beta+1)} \,.\, P^{\alpha+1,\beta}(u)$

Whence,

$K_t^{(\alpha,\beta)}(1) = \dfrac{1}{2^{\alpha+\beta+1}} \cdot \dfrac{\Gamma(t+\alpha+\beta+2)}{\Gamma(\alpha+1)\,\Gamma(\alpha+2)\,\Gamma(t+1)\,\Gamma(t+\beta+1)} \,.\, P^{\alpha+1,\beta}(u)$

In fact, we need to calculate the quantity

$$\Delta^{(\alpha,\beta)} = 2^{\in}\,\Gamma_{\alpha,\beta}\,K^{(\alpha,\beta+\in)}\,(1)$$

where $\in = \in t$ = res(*t*)(mod 2) and

$$\Gamma_{\alpha,\beta} = \int_{-1} W_{\alpha,\beta}(u)\,du = \frac{2^{\alpha+\beta+1}\,\Gamma(\alpha+1)\,\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}$$

By substitution , we obtain

$$\Delta^{(\alpha,\beta)} = \frac{\Gamma(\beta+1)\,\Gamma([t/2]+\alpha+\beta+\in+2)\,\Gamma([t/2]+\alpha+2)}{\Gamma(\alpha+\beta+2)\,\Gamma(\alpha+2)\,\Gamma([t/2]+1)\,\Gamma([t/2]+\beta+\in+1)}$$

The following specialization is the most important in the sequel.

THEOREM *:* Let $m \in$ N; m >= 1; and let

$\Delta_k(m,t) = \Delta_t^{(\alpha,\beta)}$, $\alpha = \partial m - \delta - 2/2$, $\beta = \delta - 2/2$

where, $\delta = [K : R]$. Then,

$$\Delta^{(m,t)}{}_R = \binom{m+t-1}{m-1}$$

and

$$\Delta c^{(m,t)} = \binom{m+[t/2]-1}{m-1}\binom{m+[t+1/2]-1}{m-1}$$

and

$$\Delta_H^{(m,t)} = \binom{2m+[t/2]-2}{2m-2}\binom{m+[t+1/2]-1}{2m-2}$$

Proof.

$\Delta_K^{(m,t)} = \dfrac{\Gamma(\delta/2)\,\Gamma([t/2]+\delta m/2+\in)\,\Gamma([t/2]+\delta m/2-\delta/2+1)}{\Gamma(\delta m/2)\,\Gamma(\delta m/2-\delta/2+1)\,\Gamma([t/2]+1)\,\Gamma([t/2]+\delta/2+\in)}$

If K = R, i.e. δ = 1, then

$\Delta_R^{(m,t)} = \dfrac{\Gamma(1/2)\,\Gamma([t+\in/2+m/2]\,\Gamma([t-\in/2]+m/2+1/2)}{\Gamma(m/2)\,\Gamma(m/2+1/2)\,\Gamma([t-\in/2]+1)\,\Gamma([t+\in/2]+1/2)}$

$= \dfrac{\Gamma(1/2)\,\Gamma([t+\in/2+m/2]\,\Gamma([t-\in/2]+m/2+1/2)}{\Gamma(t+\in/2+1/2)\,\Gamma(m/2)\,\Gamma([m/2]+1/2)} \dfrac{1}{(t-\in/2)!}$

Because of the classical formula

$\Gamma(u+k) = \Gamma(u)\prod(u+i)$, $k \in$ N,

we get,

$\Delta_R(m,t) = \dfrac{\prod(m/2+i)\prod(m+1/2+i)}{\prod(1/2+i)\,(t-\in/2)!}$

$= \dfrac{\prod(m+i)\prod(m+1+i)}{(t+\in-1)!!\,(t-\in)!}$

= m(m + 1) ……… (m + t -1 / m - 1) .

For K = C or H, i.e. δ = 2 or 4, X = δ / 2 , formula becomes

$\Delta_K(m,t) = \dfrac{(t+\in/2+X^m-1)!\,(t-\in/2+X^m-X)!}{}$

$(X^m - 1)!\ (X^m - X)!\ (t - \in/2)!\ (t + \in/2 + X - 1)!$

$= (X^m - X)!\ (t - \in/2 + X^m - X)!\ (t - \in/2 + X^m - 1)!$

$(X^m - 1)!\ (X^m - X)!\ (t - \in/2)!\ (t + \in/2 + X - 1)!$

$= \quad 1$

$\_\_\_\_\_\_\_\_\_\_ (X^m + [t/2] - X)(X^m + [t+1/2] - 1)$

$(X^m - X + 1) \ldots\ldots (X^m - 1)\quad (X^m - X \quad)(X^m - X \quad)$

The product in the denominator is 1 for X = 1 and 2*m* - 1 for X = 2.

INTEGRATION OF ZONAL FUNCTIONS

Here we derive some integration formulas we have used in the main text. We denote by σ\` the Lebesgue measure (area) on the sphere $S^{q-1} = S(R^q)$ induced by the standard Lebesgue measure (volume) in R^q: The normalized measure on S^{q-1} will be denoted by σ, so that

$\sigma = \sigma\text{\`} / \text{Area}(S^{q-1})$

From now on for any measure u, we use the short notation

$\int f.\ du$

meaning the integration over the support of u or a set Z C suppq.

THEOREM : Let f be a continuous function on [-1; 1]. Then for all $x \in S^{q-1}$

$$\int f(\langle x , y\rangle)\, d\sigma(y) = \int_{-1} f(u)\, \Omega(u)\, du$$

Proof. Consider the decomposition R^q = Span(x) © L; so that L , x and $y = \xi_1 x + z$; $z \in$ L: Let σ\` be the area on the unit sphere S(L) == $S^{q-2}$ induced by σ\`. Then

$$d\sigma\text{\`}(y) = (1 - \xi_1^2)^{q-3/2}\, d\xi_1\, d\sigma\text{\`}(z)$$

As a result,

$\int f(\langle x , y\rangle)\, d\sigma(y) = \int_{-1} f(\xi_1)\,(1 - \xi_1^2)^{q-1/2}\, d\xi_1 = k\text{\`} \int_{-1} f(u)\, \Omega_m(u)\, du$

where k and k\` are some coefficients. Actually, k\` = 1 since the measure σ and the weight Ω_m are both normalized.

Now we obtain a modification of regarding to the projective situation. The latter means that the integrand only depends on | <x,y> | or, equivalently, on | <x,y> $|^2$: We start with a multi-dimensional counterpart .

LEMMA *:* Let 2 <= l<=· q- 2: The measure σ\` is the product

$$d\sigma\text{\`}(y) = (1 - p^2)^{1/2 - 1}\, p^{q-l-1}\, dp\, d\text{\`}\sigma_{l-1}(z\text{\`})\, d\sigma\text{\`}_{q-l-1}(w\text{\`})$$

where $y = [Ci]^q_1 \in S^{q-1}$, $z = [Ci]^l_1$, $w = [Ci]^q_{l+1}$, q = ||wk|| and e\`i-1 is the measure (area) induced on the sphere S^{i-1} C R^i, 2 <= i <= q - 1:

Proof. There is the diffomorphism

$y \to (p , z\text{\`} , w\text{\`})$; $0 < p < 1$; $z\text{\`} \in S^{l-1}$; $w\text{\`} \in S^{q-l-1}$;

its inverse diffeomorphism is

$y = [\sqrt{1 - p^2}\, z\text{\`}]$

$[\quad pw\text{\`} \quad]$

Denote by $v_1; : : : ; v_{l-1}$ and $\psi_1; : : : ; ' \psi_{l-1}$ the spherical coordinates on S^{l-1} and S^{q-l-1} respectively, so that $(\xi1; : : : ; \xi q) \to (p; v_1; \ldots ; v_{l-1}; \psi1; : : : ; \psi_{q-l-1})$. The corresponding Jacobi matrix is

$[\ -p/\sqrt{1 - p^2}\, z\text{\`} \quad \sqrt{1 - p^2}\, z\text{\`} \quad 0 \quad]$

$[\quad w\text{\`} \quad 0 \quad pW \quad]$

where,

$Z = [\ \partial\xi i\text{\`} / \partial\partial k\ ]$, 1 <= i <= l , 1 <= k <= l – 1,

and

$W = [\ \partial\xi i\text{\`} / \partial k\ ]$, 1+1 <= i <= q , 1 <= k <= q - l – 1,

The first column is orthogonal to the others since

$\sum \xi i\ \partial\xi i\text{\`} / \partial k = ½\, \partial\, \xi i / \partial vk\, (\sum\xi i^2) = 0$

and, similarly ,

$\sum \xi i\text{\`}\ \partial\xi i\text{\`} / \partial k = 0$

Since the Norm of the first column is

$\sqrt{p^2 / 1-p^2 + 1} = (1 – p^2)^{-1/2}$,

the corresponding Gram matrix is

G =[$(1 – p^2)^{-1}$ 0 0]

[0 $(1 – p^2)Z`Z$ 0]

[0 0 $p^2 W`W$]

However, Z and W are the Jacobi matrices of the transformations (v1; : : : ; v_{l-1}) → ((`1; : : : ; (`l) and (ψ1; : : : ; 'ψq- l - 1) → ((`l+1; : : : ; (`q) respectively. Therefore,

$d\sigma`(y)=\sqrt{detG}\ dpv_1…dv_{l-1}d'_1 : : d'_{q-l-1} = 1 - p^2)^{l/2\ -1}\ p^{q-l-1} dpde`_{l-1}(z`)de`_{q-l-1}(w`)$

REMARK : Formula is also valid for l = 1. The measure σ`0 on the 0-dimensional unit sphere $S^0$ = {-1; 1}C R is such that $\sigma`_0(1) = \sigma`_0(-1) = 1$.

Below we apply Lemma to l = ∂; q = ∂m with ∂ = [K : R] and K = R; C or H. Then $S^{q-1} = S(E_R) = S(E)$ where E is a m-dimensional (m >= 2) right linear Euclidean space over K and ER is the realification of E.

THEOREM : Let ø be a continuous function on [0; 1]. Then for all x ∈ S(E)

$$\int ø (|<x , y>|^2)\ d\sigma (y) = \int_{-1} ø (1+v/2)\ \Omega_{\alpha,\beta}(v)\ dv$$

where,

$$\alpha = \delta m – \delta – 2 / 2 ,$$

$$\beta = \delta – 2 / 2$$

Proof. Consider the coordinate system in E = K^m with the first basis vector x ∈ S(E). If

y = $[(`i]^{\delta m}{}_1$ then <x; y> = $(`_1 \in K^m\ R^\delta$ so, z = $(`_1$; w = $[(`i]^m{}_2 \in K^{m-1} == R^{\delta m-1}$ in notation of Lemma . Applying this lemma for δ = 2; 4 and Remark for δ = 1 we obtain

$\int ø (|<x , y>|^2)\ d\sigma (y) = \int_{-1} ø (| (`_1 |^2)\ d_\sigma(y)$

$= k \int d\sigma`_{\delta-1} \int d\sigma`_{\delta m - \delta-1} \int_1 ø (1-p^2) (1-p^2)\delta/2 -1\ p_{\delta m - \delta-1}$ dp

$= k_1 \int_1 ø (1-p^2)\ p^{2\alpha+1} (1-p^2)^\beta dp$

where k and k_1 are some coefficients. By substitution $1-p^2= 1/ 2(1 + v)$;

$\int ø (|<x , y>|^2)\ d\sigma (y) = k_2 \int_{-1} ø ((1 + v)/2) (1 – v)^\alpha (1+v)^\beta dv$

$= k_3 \int_{-1} ø ((1 + v)/2)\ \Omega_{\alpha,\beta}(v)\ dv$

with some coefficients k_2 and k_3. In fact, we get $k_3 = 1$ taking ø = 1 as before.

COROLLARY : For t ∈ N the quantity

$$\mathfrak{y}_k (m,t) = (\int | <x,y> |^{2t}\ d\sigma(y))^{-1} , x \in S(e)$$

is independent of x , namely,

$$\mathfrak{y}_R (m,t) = \frac{(2t + m -2) !!}{(m -2) !! (2t – 1) !!}$$

and

$$\mathfrak{y}_C (m,t) = \binom{t + m -1}{m -1}$$

and

$$\mathfrak{y}_H (m,t) = 1/ (t+1) \binom{t + 2m -1}{2 m -1}$$

Proof :

$\int (|\langle x, y\rangle|^{2t})\, d\sigma(y) = \int_{-1} (1+v/2)^t\, \Omega_{\alpha,\beta}(v)\, dv$
$= 1/2^t\, \Gamma_{\alpha\beta}$

$\Gamma_{\alpha\beta + t}$
$= \Gamma(\beta + 1)\, \Gamma(t + \alpha + \beta + 2)$

$\Gamma(\alpha + \beta + 2)\, \Gamma(t + \beta + 1)$

i.e;

$\upsilon_K(m,t) = \Gamma(\delta/2)\, \Gamma(t + \delta m/2)$

$\Gamma(\delta m/2)\, \Gamma(t + \delta/2)$

If K = R, i.e. δ = 1, then

$\upsilon_K(m,t) = \Gamma(1/2)\, \Gamma(t + m/2)$

$\Gamma(m/2)\, \Gamma(t + 1/2)$

$= (m + 2t - 2)!!$

$(m - 2)!!\ (2t - 1)!!$

If K = C or H, i.e. δ = 2 or 4, then

$\upsilon_K(m,t) = (t + \delta m/2 - 1)!$

$(\delta m/2 - 1)!\, (t + \delta/2 - 1)!$

$= t! \quad (t + \delta m/2 - 1)$

$(t + \delta/2 - 1)! \quad (\delta m/2 - 1)$

The latter fraction is equal to 1 or 1 /*t*+1 if δ = 2 or 4 respectively.

Note that

$\upsilon_K(m,0) = 1$

$\upsilon_K(m,1) = m$

irrespective to k.

COROLLARY : **For all x ∈ E the Hilbert Identity**

$\langle x, x\rangle^t = \upsilon_K(m,t) \int |\langle x,y\rangle|^{2t}\, d\sigma(y)$

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