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A New Paradigm for Thinking about Mathematics

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Mathematics has long been viewed as the pinnacle of the rationalist tradition. In this chapter I will argue that a new paradigm for discourse about mathematics has begun to emerge. I begin by situating this new paradigm in the context of other post-modern attacks on rationalism. The historical and epistemological roots of the problems with rationalism in mathematics are then explored. The increasing role of the computer in mathematics will be seen as contributing to the downfall of earlier concepts of proof but, as in so many other disciplines into which it has entered, the effect of the computer has been contradictory -- pushing mathematical practice both in more formal and more intuitive directions.

Rationalism has come under attack in our post-modern era. It has been criticized by hermeneutic critics such as Packer & Addison (1989). Among the differences, they cite in their critique, between the rationalist perspective and the hermeneutic (or interpretive) perspective, four dimensions of difference are salient:

Rationalism Hermeneutics Ground of Knowledge Foundation provided by axioms starting place provided and principles by practical understanding: articulated and corrected.

Character of Explanation Formal, syntactic reconstruction Narrative accounts – a of competence reading of the text Relationship to Researched Detachment: abstraction from Familiarity with practices: context participation in shared culture Justification of Explanation Assess correspondence with Consider whether interpretation knowledge of competent person uncovers an answer to its motivating concern.

Historians and Sociologists such as Latour (1987) have critiqued the traditional scientific method and emphasized that science can only be understood through its practice. Feminist critics such as Gilligan (1986) and Belenky et al (1986) have criticized rationalism from a psychological perspective. They show how the formal knowledge of rationalist procedures has created a "separate knowing" which has alienated many and women in particular. They propose a revaluing of personal knowledge and what they call "connected knowing" - a knowing in which the knower is an intimate part of the known.

The construction of the known through the interpretation of the knower is fundamental and unavoidable for non-alienated understanding. The concept of "connected knowing" will be explored in more detail in chapter IV, in the discussion of "concretion."

While these critiques of the rationalist (and empiricist) traditions have made serious inroads into the hegemony of the dominant epistemology, the calls for interpretive frameworks have largely focused on the social sciences and to a lesser degree on the natural sciences. To a great extent, mathematics has still escaped the full glare of this critique. However, this is beginning to change -- we are witnessing the emergence of a new paradigm for thinking about the character of the mathematical enterprise.

In order to understand the nature of the paradigm shift, it will be helpful to situate this discussion in the history of mathematical epistemology. In the next section, I will sketch the development of mathematicians' thinking about their own practice. We shall see that mathematics, which for so long was seen to be about the search for truth, or the discovery of the properties of Platonistic entities, derived its meaning from the "reality" of the Platonic "world".

Crises in mathematics arising in the 19th and 20th centuries created problems for this view. Two classes of responses to these crises have emerged. The first, and earliest, a formalist agenda whose aim was to eliminate ambiguity, interpretation and meaning from mathematical symbols.

This response, which arose from the desire to rescue mathematics from contingency and secure Or as Gilligan refers to it - the "web of connectedness". The indomitability of its foundations greatly influenced the development of pedagogy for mathematical instruction (see e.g. Kleiner, 1990). However, a second response, motivated by the preservation of meaning in mathematical statements, has begun to take hold. This latter view sacrifices the idea of a realm of mathematical entities, and places the emphasis on the construction of these entities by a community of mathematical meaning makers. In so doing, it replaces the centrality of truth and validity in mathematics by the interpretation and negotiation of meaning. Seen from this perspective, we shall see that mathematical proof, which was seen to be the foundation of mathematical validity, becomes instead a method for connecting new knowledge to our personal knowledge web and thus imbuing it with meaning.

EPISTEMOLOGY OF MATHEMATICS - A SHORT HISTORY

While mathematical activity has been going on as far back as is recorded, the idea of mathematical demonstration apparently only emerged in about 600 B.C.E. Thales of Miletus is credited with bringing geometry from Egypt to Greece. Although the Egyptians had an implicit geometry in their surveying practice, as far as we know they did not prove any geometrical theorem. Thales is credited with having proved a number of elementary theorems including that the base angles of an isosceles triangle are congruent. By about three hundred years later, Euclid had axiomatized plane geometry so that all theorems could be deduced from five "self-evident" postulates.

THE ONTOLOGICAL STATUS OF MATHEMATICAL OBJECTS

Since the beginning of this proof era of mathematics, mathematicians have also concerned themselves with the epistemology of mathematics. What kinds of things are mathematical objects? What justifies mathematical arguments? Are mathematical truths *a priori* (prior to experience) or are they derived from our everyday experience?

Plato (see Hackforth, 1955) gave one of the first answers to these questions and to this day most mathematicians regard themselves as Platonists with respect to the reality of mathematical objects (see Davis and Hersh, 1981). Plato's theory was that the world of everyday objects is just a shadow world. Beyond the shadows lies a real world of transcendent objects in which live "ideal forms". Ideal forms are unique, immutable, and embody timeless truths. So, for example, while in the passing shadow world we may see many instances of stones, in the transcendent real world, there is one stone form of which all everyday stones are shadows. In this real world, mathematical objects are first class citizens. In the everyday world, one can find two apples, but the concept-object "two" does not exist at all except in the world beyond the senses. Central to this view is the notion that ideal forms are apprehensible, but unlike everyday objects which are apprehended through the senses, concept-objects must be apprehended by a process of pure ratiocination, a direct intellectual connection to the transcendent world.

This view has been elaborated on through the rationalist tradition and falls under the heading of what is now called Mathematical realism. One consequence of this view is that our intuitive notions of mathematical objects must be coherent since they reflect the real objects in the transcendent world. Mathematical knowledge is certain according to this view and the foundation of its certainty lies in the real relationships that obtain between real mathematical entities.

One implicit consequence of this view is that mathematics is true in the everyday world since that world is derived from the ideal world. Thus Euclidean Geometry was thought to be necessarily true of the world.

Throughout the middle Ages, mathematicians tried to derive Euclid's fifth postulate (the so-called "parallel postulate") from the other four. The fifth postulate was not as self-evident as the other four and this was bothersome to the sense of necessity that geometry was supposed to have. The fifth postulate however resisted all attempts to derive it from the other four or from yet another more "self-evident" postulate. Finally, in the nineteenth century, Lobachevsky, Bolyai, and Riemann were able to construct models of geometry in which Euclid's first four postulates were true, but the fifth was not. In Riemannian geometry, through a point not on a given line, one cannot draw a line parallel to the given line. The sum of the angles of a triangle is always more than two right angles, and the ratio of a circle's circumference to its diameter is always less than pi.

If however, Euclidean geometry expressed truths about the world, then these alternative geometries would have to be proved false. Naive Platonism was dealt a blow when relative consistency theorems were proved showing that if Euclidean geometry is consistent so are the alternative geometries.

The meaning of the statement "Euclidean geometry is true," epistemologists had said, is that the postulates are true when the geometrical terms are interpreted in normal fashion. But what does it mean to interpret the terms in normal fashion? For example what does the term "straight" mean in "straight line"? One way of telling whether a line segment is straight is to see if you can find any measuring rod which is shorter and connects its end-points. Or one could sight along it - this would mean that straight is the path light takes, or one can define it as the path on which a stretched rope lies when it is under high tension. Another way to define it is to say that a straight line is the shortest distance between two points. Which of these definitions captures the essence of straightness? At the time, all criteria seemed to agree, so this question wasn't carefully addressed and Euclidean geometry was assumed to be true under standard usage of the word "straight".

When Einstein demonstrated the theory of general relativity, a further question arose as to the truth of Euclidean geometry. The truth of Euclidean geometry then appeared to depend on empirical observations - how does light travel, what does the rod measure? When Einstein showed that light does not travel in a Euclidean line, did he show that Euclidean geometry is false?

Some mathematicians do indeed take this view. According to them, our notion of straightness is empirically based and comes from our experience with light and rods.

Hence relativity (by showing that light triangles have more than 180 degrees) proves that space is Riemannian not Euclidean.

But there is another view that says, no, straightness is defined in terms of shortest distance where distance is determined by an Euclidean metric. Under this view, space is not Riemannian, it is still Euclidean, it is only that gravity (like temperature, pressure, refractive medium) bends rods and light so that they are no longer straight.

Einstein himself espoused the former view as expressed in his famous quote: "As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality".

THE FORMALIST RESPONSE TO THE CRISIS

To escape such questions about the truth of geometry, some mathematicians began to tighten up Euclid's postulates so that they could be put on a more secure footing -- turned into a formal system. Thus was born the *logicist* program to reduce mathematics to logic. According to this view, Euclidean and Riemannian geometry are equally valid -- they state nothing more than: if their postulates are true then their theorems follow logically. Some such as Poincaré (1908) carried this further into so-called "conventionalism" and said that the truth of Euclidean geometry vs. Riemannian geometry was a matter of convention -- how we stipulate the definitions of our language.

The logicist program sought to formalize mathematics so that there would be a purely syntactic procedure for determining the truth of a mathematical statement. To do so would require that all referent terms in a mathematical expression remain uninterpreted, without meaning apart from their formal stipulations. A term in such a system is a "dead" object defined only in terms of its relations to other dead objects. A proof, in this view, contains all the information necessary for believing and asserting a theorem.

Parallel to the controversy about alternative geometry, another group of mathematicians was engaged in trying to formalize arithmetic. Peano and Dedekind had provided axiomitizations of the natural numbers and in 1879 Frege was the first to articulate the aim of reducing all of mathematics to logic (Frege, 1953). However, Frege's formulation relied on the intuitive notion of class. Russell showed that this formulation was inconsistent. In the intuitive notion of class was the idea that any describable collection could be a class. In particular, the class consisting of all classes is just another class. Here already the intuitive notion becomes muddy since at once we have this class being as big as imaginable and at the same time it is just one member of the large collection of classes which are its members. Russell formalized this paradox by considering the class of all classes that are not members of themselves. Call this class C.

Is C a member of itself? If it is, then by the definition of C, it's not. If it's not, then by the definition of C, it must be. Russell resolved this paradox by assigning types to classes and classes that contain other classes are of higher type than their members. But the cost of this move was high. The intuitive notion of class which seemed perfectly innocent had to be replaced by the unintuitive notion of class of type "n". This was another blow to Platonism. What happened to the ideal form of class? Are we to suppose that there are infinitely many ideal forms corresponding to a class of each type?

Logical positivists (e.g. Carnap, 1939; Ayer, 1946) responded to this crisis of meaning by adopting an extreme form of conventionalism which said that mathematical statements do not need any justification because they are true by fiat, by virtue of the conventions according to which we stipulate the meanings of the words we choose in Referring to the significance of this paradox, Frege said: “Arithmetic trembles”. Mathematics. In so doing, they removed mathematics both from doubt and from any possibility of making personal connections to, and meaning from mathematical objects.

Others, (e.g. Kitcher, 1983; Putnam, 1975) who saw that the procedure for choosing axioms for set theory and arithmetic was one of making hypotheses, deducing consequences and then evaluating these consequences, moved in the opposite direction and began to see mathematical process as much more akin to scientific hypothetic- deductive process rather than in a removed realm all its own.

THE FAILURE OF FORMALISM

The logicist dream of reducing mathematics to logic and Hilbert's derivative program to prove mathematics consistent were dealt a final blow by the results of Gödel (1931). Gödel's theorems show that any formal system large enough to incorporate basic arithmetic cannot be proven consistent within that formal system. Furthermore, if the system is consistent then there are statements in the system which are true but unprovable.

Since Gödel, further unclarity in the basic notion of the integers emerged when specific hypotheses such as the continuum hypothesis were proven independent of the standard axiomatizations of arithmetic. Mathematicians were confronted with the In the last century, old arguments about the reality of the continuum, which date back to Zeno in the early 5th century BCE, were revived. Why is it permissible to divide space up into infinitely small pieces? What legitimates making infinitely many choices in finite time? Even if we can make infinitely many choices, can we make an uncountably infinite number of choices? These questions have a crucial bearing on what the character of the so-called "real" numbers is. Mathematical Constructivists such as Kronecker asserted that the real numbers were not real at all but a convenient fiction useful for some human purposes. (As in his famous quote: "God created the integers, all the rest were created by man.") Recently, some mathematicians, notably Brian Rotman (1993), have done Kronecker one better and questioned the legitimacy of the integers themselves. By what criterion are we allowed to assert that we can count forever? The intuitions that allow us to say that addition is, for example, commutative are formed from experience with small numbers -- what legitimates the extrapolation to numbers infinitely far away? Here we see clearly that making mathematics requires acts of imagination. Which imaginings or, if you will, "mathematical dreams" are we entitled to and which are we to say are irrational? (or ir-rational?) Which asserts that there are no infinities "larger" than the integers yet "smaller" than the reals.

situation of having to explicitly choose whether to adopt an axiom by criteria such as plausibility, simplicity, and productivity in generating good mathematics.

These developments in formal mathematics permanently halted the logicist agenda. It would seem that mathematics is beyond logic and its truths are not formalizable.

POST-MODERN VIEW OF MATHEMATICS AND SCIENCE

In the latter half of the twentieth century, developments in the philosophy and history of science have tended to push mathematicians into seeing mathematics as more continuous with natural science. Works by Popper (1959), and more decisively by Kuhn (1962) have shown that the progress of science is not linear and hypothetico-deductive (see Hempel, 1963) as the logical positivists have claimed, but rather science proceeds by revolution, i.e. by changing the meaning of its basic terms. New theories are not incremental modifications of old theories, they are incommensurate in that what they posit as the basic entities of the world are fundamentally incompatible. Furthermore, says Kuhn, by presenting science as a deductive system devoid of history, positivists have robbed us of the fundamental character of science – the negotiation and construction of its basic entities. Mathematicians such as Polya (1962) and Lakatos (1976) have shown that mathematical development has been similarly mischaracterized. By placing such a strong emphasis on mathematical verification and the justification of mathematical theorems after their referent terms have been fixed, mathematics literature has robbed our mathematics of its basic life. Mathematics, according to Lakatos, is a human enterprise.

Advances in mathematics happen through the negotiation of a community of practitioners. Moreover, the development of mathematical proofs is not linear, but rather follows the "zig-zag" path of example, conjecture, counter-example, revised conjecture or revised definition of the terms referred to in the conjecture. In this view, mathematical meaning is not given in advance by a transcendent world, nor is it stipulated in an community of practitioners and given meaning by the practices, needs, uses and applications of that community.

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