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REVIEW ARTICLE

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Review of the Elements of Intersection Theory for Two Dimensional Schemes

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We review here the elements of intersection theory for two dimensional schemes. In mathematics, intersection theory is a branch of algebraic geometry, where subvarieties are intersected on an algebraic variety, and of algebraic topology, where intersections are computed within the cohomology ring. The theory for varieties is older, with roots in Bézout's theorem on curves and elimination theory. On the other hand the topological theory more quickly reached a definitive form.For a connected oriented manifold M of dimension 2n the intersection form is defined on the nth cohomology group (what is usually called the 'middle dimension') by the evaluation of the cup product on the fundamental class

$$[M] \in H_{2n}(M, \partial M)$$

$$\lambda_M : H^n(M, \partial M) \times H^n(M, \partial M) \to \mathbb{Z}$$

$$\lambda_M(a, b) = \langle a \smile b, [M] \rangle \in \mathbb{Z}$$

This is a symmetric form for n even, in which case the signature of M is defined to be the signature of the form, and an alternating form for n odd. It is possible to drop the orient ability condition and work with\mathbb{Z}_2 coefficients instead.

These forms are important topological invariants. For example, a theorem of Michael Freedman states that simply connected compact 4-manifolds are (almost) determined by their intersection forms up to homeomorphism.

By Poincaré duality, it turns out that there is a way to think of this geometrically. If possible, choose representative n-dimensional submanifolds A, B for the Poincaré duals of a and b. Then λM(a, b) is the oriented intersection number of A and B, which is well-defined because of the dimensions of A and B. Let 0 be a Dedekind domain, K its field of fractions and C a scheme over O. We will call C a curve over 0 if C is an integral two dimensional scheme which is proper and flat over 0 and K is algebraically closed in the field of fractions of C. We say that C is regular if all its local rings are regular. It is a theorem of Lichtenbaum that C is the projective over 0, see [Li].

A well-working machinery of intersecting algebraic cycles V and W requires more than taking just the set-theoretic intersection of the cycles in question. Certainly, the intersection V ∩ W or, more commonly called intersection product, denoted V · W, should consist of the set-theoretic intersection of the two subvarieties. However it occurs that cycles are in bad position, e.g. two parallel lines in the plane, or a plane containing a line (intersecting in 3-space). In both cases the intersection should be a point, because, again, if one cycle is moved, this would be the intersection. The intersection of two cycles V and W is called proper if the codimension of the (set-theoretic) intersection V ∩ W is the sum of the codimensions of V and W, respectively, i.e. the "expected" value.

Therefore the concept of moving cycles using appropriate equivalence relations on algebraic cycles is used. The equivalence must be broad enough that given any two cycles V and W, there are equivalent cycles V' and W' such that the intersection V' ∩ W' is proper. Of course, on the other hand, for a second equivalent V" and W", V' ∩ W' needs to be equivalent to V" ∩ W".

For the purposes of intersection theory, rational equivalence is the most important one. Briefly, two r-dimensional cycles on a variety X are rationally equivalent if there is a rational function f on a (k+1)-dimensional subvariety Y, i.e. an element of the function field k(Y) or equivalently a function f : Y → P1, such that V - W = f-1(0) - f-1(∞), where f-1(-) is counted with multiplicities. Rational equivalence accomplishes the needs sketched above.

Let C be a regular curve over 0 and let D and E be two Weil divisors of C. We say that D and E meet properly if the intersection of their support is a finite number of closed points of C. We say that a divisor is prime if it corresponds to an integral subscheme. Suppose that D and E meet properly and are both prime divisors. Let $x \in C$ be contained in both of their supports. Let

$$(D,E)_x = \dim_{k(x)} (O_x/f,g),$$

Where O_x is the local ring of C at x, k(x) is the residue field, f (resp. g) is the local equation of E (resp. E) at x. We extend this definition by linearity to any two divisors D and E which meet properly. We'll call $(D,E)_x$ the local pairing of D and E at x.

For the rest of this section let O be a discrete valuation ring with maximal ideal p, and let C be a regular curve over 0. We review here the work of Lichtenbaum [LiJ and Shafarevich [Sh] concerning an intersection pairing on C. We say that a Weil divisor of C lies over p if its support lies over p. Let D and E be Weil divisors of C which meet properly and suppose that either D or E lies over p. Let

$$(D,E) = \sum_x (D,E)_x \dim_{k(p)} k(x).$$

Here the summation runs over closed points x that are in the supports of both D and E. We'll call (,) the L-S pairing. It is clearly additive whenever it's defined.

Let D be a Weil divisor of C which lies over p. Let f be a non-zero rational function on C whose associated divisor (f) meets D properly. Prop Is (D,(f)) = 0.

Proof:

Consequently we may extend the definition of (D/E) to arbitrary divisors D and E of C such that at least one of them lies over p. **§2** We review here Arakalov's theory of intersections, [Ar].

In this section K will be a number field, 0 its ring of integers, and C a regular curve over 0. Let n = [K:Q], and let $\phi_1,\ldots,\phi_n$ be the set of embeddings of K into C.

Let C_i denote the Riemann surface associated to $C \otimes_{\phi_i} \mathbb{C}$. Let $d\mu_i$ be a positive (1,1) form on C_i with

$$\int_{C_i} d\mu_i = 1.$$

Let V be an n dimensional real vector space with basis $v_1,\ldots,v_n$. Let Div^V C be the group Div $C \oplus V$, the group of Arakelov divisors of C. Let f be a non zero rational function on C, and let

$$v_i(f) = \int_{C_i} \log |f| d\mu_i,$$

where $|f|$ denotes the absolute value of the function on C_i Here (f) refers to the principal Weil divisor of C associated coming from f. Let v(f) be the vector in V whose i^{th} coordinate (with respect to the basis $\{V_i\}$ is v_i (f). Let

$$(f)_A = (f) + v(f) \in \mathrm{Div}^V C.$$

Here (f) refers to the principal weil divisor of C associated to f. We call $(f)_A$ the Arakelov divisor associated to f, we say that $(f)_A$is a principal Arakelov divisor. Let Pic^VC be the quotient of Div^VC by principal Arakelov divisors. We call Pic^VC the group of Arakelov divisor classes.

Let $D_i = \sum n_p \cdot P$ be a Weil divisor of C_i. We call a function G on C_i the Green's function for D_i if:

a) G is a smooth non negative function on C_i away from the support of D_i.

b) $$-\frac{1}{2\pi}\Delta \log G\, dx\, dy = (\deg D) d\mu_i,$$

here z = x + iy is a local parameter for C^,

c) In a neighborhood of a point P we have:

$$G(z) = |z - z(p)|^{n_P} u(z),$$

where z is a local parameter, and u(z) is a non vanishing smooth function.

d) $$\int_{C_i} \log G\, d\mu_i = 0.$$

Prop 2 (Arakelov) Given a divisor D on C_i, a Green's function for D exists and is unique.

Let $D_i = \Sigma n_p \cdot P$ be a divisor of C_i, let G be the Green's function of D and let $E_i = \Sigma m_Q \cdot Q$ be another divisor of C_i whose support has no points in common Proposition 3 (Arakelov): Wo have with the support of D_i. Let

$$[D_i, E_i] = -\sum_Q m_Q \log G(Q).$$

$$[D_i, E_i] = [E_i, D_i].$$

Let D and E be Weil divisors of C which meet properly. Let ϕ_i be an embedding $K \to \mathbb{C}$. The divisor D (resp. E) of C determines a divisor D_i (resp. E_i) of

C_i. Let $[D,E]_{\phi_i} = [D,E]_i = [D_i, E_i]$. Let

$$[D,E] = \sum_x (D,E)_x \log |k(x)| + \sum_{i=1}^{n} [D,E]_i,$$

where the first summation runs over closed points x of C which are contained in the supports of both D and E, and the second summation runs over embeddings $\phi_i: K \to C$. We extend the definition of (,) further by letting

$$[D, \sum_i a_i v_i] = d \sum_i a_i,$$

$$[\sum_i a_i v_i, \sum_j b_j v_j] = 0$$

Here D is a Weil divisor of C which induces a divisor of degree d on $C \otimes K$, andare elements of V. $\sum a_i b_i, \sum b_j v_j$ Proposition 4 (Arakelov): Let D be a Weil divisor of C and let f be a non zero rational function such that (f) and D meet properly. We have $[D, (f)_A] = o$.

We obtain a symmetric bi-additive real valued pairing [,] on $Pic^V C$.

Let p be a maximal ideal of 0 and let 0 be the localization of 0 at p. Let D and E be Weil divisors of C and assume that D lies over p. Then $C \otimes O_p$ is a regular curve over O_P and the divisors D and E induce divisors on $C \otimes O_p$, which we'll also call D and E. We have

$$[D,E] = (D,E) \log |k(p)|.$$

In this section we study the behaviour of the pairings (, $)_x$ with respect to blow-ups and base change. We define the notion of the pullback of a divisor, prove a few facts about it, and use them to obtain several formulas we'll need later on. Unless otherwise indicated, a blow-up of a scheme has as its center a closed point. In abstract algebra, a Dedekind domain or Dedekind ring, named after Richard Dedekind, is an integral domain in which every nonzero proper ideal factors into a product of prime ideals. It can be shown that such a factorization is then necessarily unique up to the order of the factors. There are at least three other characterizations of Dedekind domains which are sometimes taken as the definition: see below.

Note that a field is a commutative ring in which there are no nontrivial proper ideals, so that any field is a Dedekind domain, however in a rather vacuous way. Some authors add the requirement that a Dedekind domain not be a field. Many more authors state theorems for Dedekind domains with the implicit proviso that they may require trivial modifications for the case of fields.

An immediate consequence of the definition is that every principal ideal domain (PID) is a Dedekind domain. In fact a Dedekind domain is a unique factorization domain (UFD) iff it is a PID.

Throughout this section 0 will be a Dedekind domain such that: for all maximal ideals p of 0, k(p) -the residue field, is finite. Let K be the field of fractions of 0. Let K' be a finite extension of K, and 0' the integral closure of 0 in K'. Let C be a regular curve over 0.

Proposition 5 : $C \otimes O'$ is a curve over 0'.

Proof : All we must check is that $C \otimes O'$ is integral.

Since C is regular, $C \otimes K$ is regular, hence smooth. Therefore $C \otimes K'$ is smooth. Since K is algebraically closed in K(C), $C \otimes K'$ is connected. Therefore $C \otimes O' \otimes K'$ is integral. Since $C \otimes O'$ is flat over 0', it must be irreducible. Since C is projective over 0, $C \otimes O'$ is projective over 0*. Let $C \otimes O' \subset \mathbb{P}^n_{O'}$, defined by some ideal I. If $C \otimes O'$ is not reduced, then $I = J^{(n)}$- the n^{fc}^ symbolic power of some prime ideal J. Tensoring with K' we see that $C \otimes O' \otimes K'$ is not reduced, a contradiction. Therefore $C \otimes O'$ is integral.

Let C_1 be the normalization of $C \otimes O'$, so C_1 is a curve over 0'. Let C' be a regular curve over 0' which is obtained from C_1 by a finite sequence of blow ups and normalizations. By the work of Abhyankar, see [Ab] or [Lip], such schemes exist. Let

$$\pi_1: C_1 \to C;$$

$$\rho: C' \to C_1; \quad \pi: C' \to C$$

be the projection maps. Let D be a Weil divisor of C. We define a divisor $\pi^* D$ of C' as follows: Let {G} be the set of prime divisors of C* whose support maps into the support of D. Let X_G be the generic point of G and let f_G be the local equation of D at $\pi(x_G)$. Then f_G induces an element $\pi^*(f_G)$ in the local ring of C' at X_G, which is a discrete valuation ring. Let n_G be the order of $\pi^*(f_G)$ with respect to this valuation. Let

$$\pi^*(D) = \sum_G n_G \cdot G.$$

Clearly this definition also applies to any map $X \to Y$ of schemes, as long as we restrict ourselves to locally principal divisors of Y and X is normal, integral and of finite type over Y.

Let D be a prime divisor of C and let $\pi^*D = \sum_i n_i D_i + F,$ where each divisor D_i of C' is prime and dominates D, and the support of F lies over a finite set of closed points of C. Let K(D) (resp. $K(D_i)$) be the field of rational functions on D (resp. D_i).

It is clear that K (D_i) is a finite extension of K(D).

Proposition 6 :

$$\sum_i n_i [K(D_i): K(D)] = [K':K].$$

Proof :

Let $\pi_1^* D = \sum n_G \cdot G,$ each G a prime divisor of C_1. Since $\pi_1: C_1 \to C$ is finite, each G must dominate D. Let x be the generic point of D and let O_x be the local ring of C at x, O_X is a discrete valuation ring.

Let R be the integral closure of O_X in K (C_1), the field of rational functions on C_1. Then R is a semi- local Dedekind domain with maximal ideals { m_G } corresponding Furthermore K(G) = R/m_G and K(D) = O_x/m. We therefore have:

to the divisors {G} above. Let m be the maximal ideal of O_x, then

$$mR = \prod_{\{G\}} m_G^{n_G}.$$

$$\sum_{\{G\}} [K(G): K(D)] n_G = [K(C_1): K(C)] = [K': K],$$

by standard results of commutative algebra, see [Bo]. To lift this result from C_1 to C' notice that $\rho: C' \to C_1$ is an isomorphism when restricted to an open set U of C_1, and $C_1 - U$ is codimension two in C_1. Therefore each divisor D_i of C' is birational via ρ to some divisor G of C_1, furthermore $n_i = n_G$.

Let D be a prime divisor of C and let $\pi^*D = \sum_i n_i D_i + F,$ as above. Let $\varepsilon: \tilde{D} \to D$ (respectively $\varepsilon_i: \tilde{D}_i \to D_i$) be the normalization. By composition we obtain a map $\varepsilon_i \circ \pi: \tilde{D}_i \to D,$ which must factor through D, obtaining a commutative diagram.

$$\begin{array}{ccc} \tilde{D}_i & \xrightarrow{\varepsilon_i} & D_i \\ \downarrow \pi_i & & \downarrow \pi \\ \tilde{D} & \xrightarrow{\varepsilon} & D. \end{array}$$

The map $\pi_i: \tilde{D}_i \to \tilde{D}$ must be finite and flat. Let $x \in \tilde{D}$ be a closed point, and let {y} be the set of points of $\tilde{D}_i$ which map to x. Let $O_{\tilde{D},x}$ (resp. $O_{\tilde{D}_i,y}$) be the local ring of D at x (resp. of $\tilde{D}_i$ at y). Let a be an element of $O_{\tilde{D},x}'$ then a induces an element of $O_{\tilde{D}_i,y}'$ which we also denote by a.

Proposition 7 : We have:

$$[K(\tilde{D}_i): K(\tilde{D})] \dim_{K(x)} (O_{\tilde{D},x}/\alpha) = \sum_{\{y\}} \dim_{K(x)} (O_{\tilde{D}_i,y}/\alpha).$$

Proof : Let R be the integral closure of $O_{\tilde{D},x}$ in $K(\tilde{D}_i)$ Then R is semi-local with maximal ideals { m_y } corresponding to the points { y }. Since π_i is finite and flat R is a free $O_{\tilde{D},x}$ module of rank $[K(\tilde{D}_i): K(\tilde{D})]$. Therefore

$$\sum_{\{y\}} \dim_{K(x)} (O_{\tilde{D}_i,y}/\alpha) = \dim_{K(x)} R/\alpha, \text{ and}$$

$$\dim_{K(x)} R/\alpha = [K(\tilde{D}_i): K(\tilde{D})] \dim_{K(x)} (O_{\tilde{D},x}/\alpha)$$

Let D be a prime divisor of C, and $\varepsilon: \tilde{D} \to D$ the normalization. Let x be a closed point of D and { y } the set of points of $\tilde{D}$ which map to x. Let $O_{D,x}$ (resp. $O_{\tilde{D},y}$) be the local ring of D at x (resp. the local ring of $\tilde{D}$ at y). Let $\alpha \in O_{D,x}'$ and denote by a the corresponding elements of $O_{\tilde{D},y}$ Proposition 8:

$$\sum_{\{y\}} \dim_{K(x)} (O_{\tilde{D},y}/\alpha) = \dim_{K(x)} (O_{D,x}/\alpha).$$

Proof: We first note that the residue field of x is finite, since D is either finite over 0 or a curve over a finite field. Similarly the residue field of every closed point of $\tilde{D}$ is finite. Let R be the integral closure of $O_{D,X}$ in K(D). Then R is a semi-local Dedekind domain with maximal ideals { m_y } corresponding to the points (y). There is some $\beta \in R$ such that $\beta R \subset O_{D,x}$ Therefore

$$R/O_{D,x} \subset R/\beta R,$$

which is finite, since every R/m_y is finite. Let M_1 be the kernel of the map $O_{D,x}/\alpha \to R/\alpha_s$ and let M_2 be the cokernel. We have a diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & m_1 \arrow[d] & \\
0 \arrow[r] & O_{D,x} \arrow[r] \arrow[d] & O_{D,x} \arrow[r] \arrow[d] & O_{D,x}/\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & R/\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R/O_{D,x} \arrow[r] \arrow[d] & R/O_{D,x} \arrow[r] \arrow[d] & m_2 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with all squares commutative and all horizontal and vertical sequences exact. By the snake lemma we obtain an exact sequence:

$$0 \to m_1 \to R/O_{D,x} \to R/O_{D,x} \to m_2 \to 0.$$

Therefore $\dim_{K(x)} m_1 = \dim_{K(x)} m_2.$ We obtain

$$\dim_{K(x)} (O_{D,x}/\alpha) = \dim_{K(x)} R/\alpha = \sum_{\{y\}} \dim_{K(x)} (O_{\tilde{D},y} \; . \;)$$

Let E be a prime divisor of C which meets D properly. Let x be a closed point of C which is contained in the supports of both D and E. Let {w} be the finite set of closed points of C' such that: (i) each w lies over x, (ii) each w lies in the support of some D_i ; (ii) each w lies in the support of π^*E Proposition 9: We have

$$[K' : K](D,E)_x = \sum_{\{w\}} \Big(\sum_i n_i D_i, \ \pi^*D\Big)_w \dim_{K(x)} k(w)$$

Proof : We have

$$(D,E)_x = \dim_{k(x)} (O_{C,x}/f,g)$$

where f (resp. g) is the local equation for D (resp. E)

in $O_{C,x}$. Clearly $O_{C,x}/f = O_{D,x}$. Therefore

$$(D,E)_x = \sum_{\substack{y \in \tilde{D} \\ y \to x}} \dim_{k(x)} (O_{\tilde{D},y}/g),$$

by proposition 8. Applying proposition 7, we obtain, for each i:

$$\begin{aligned}
[K(D_i) : K(D)](D,E)_x &= \sum_{\substack{y \in \tilde{D} \\ y \to x}} \ \sum_{\substack{z \in \tilde{D}_i \\ z \to y}} \dim_{k(x)} (O_{\tilde{D}_i,z}/g) \\
&= \sum_{\substack{z \in D_i \\ z \to x}} \dim_{k(x)} (O_{D_i,z}/g) \\
&= \sum_{\substack{z \in D_i \\ z \to x}} (D_i, \pi^*E)_z \dim_{k(x)} k(z).
\end{aligned}$$

Combining this with proposition 6, we obtain our result.

In this section we apply the results of the previous section to Arakelov's pairing [,]

Let 0 be the ring of integers of a number field K. Let K' be a finite extension of K, O' the integers of K'. Let C be a regular curve over O, and C' a regular curve over O' which is obtained from $C \otimes O'$ by a finite sequence of normalizations and blow ups. Let $\{\phi_i\}$ be the set of embeddings $K \to \mathbb{C}$, and let $d\mu_i$ be a (1,1) form on C_i, as in section 2. For each ϕ_i, let $\{\phi_{i_j}\}_j$ be ■ the set of embeddings of K' into C which induce ϕ_i.

Let C'_{ij} be the Riemann surface associated to $C' \otimes_{\phi_{i_j}} C$. We have $C_i \simeq C'_{ij}$. Employing this isomorphism and the form $d\mu_i$, we obtain a form $d\mu_{ij}$ with

$$\int_{C'_{ij}} d\mu_{ij} = 1.$$

Let V' be the real vector space with basis $\{v_{ij}\}$ and let $\mathrm{Div}^V C' = \mathrm{Div}\, C' \oplus V'$. We define a map $\pi^*: \mathrm{Div}^V C \to \mathrm{Div}^V C'$ by

$$D + \sum_i r_i v_i \to \pi^* D + \sum_{i,j} r_i v_{ij}.$$

This map clearly induces a map $\pi^*: \mathrm{Pic}^V C \to \mathrm{Pic}^V C'$. Proposition 10: For $D,E \in \mathrm{Div}^V C$ we have $[K': K][D,E] = [\pi^*D, \pi^*E]$

Proof : One immediately reduces to the case that D and E are Weil divisors and meet properly. Let $\pi^*D = \sum n_k D_k + F$ with each D_k dominating D via π and F lying over a finite set of closed points of C. It is easily seen that

$$[K': K]\sum_i [D,E]_i = \sum_{i,j} [\sum_k n_k D_k, \pi^*E]_{ij}$$

Proposition 9 : shows that

$$[K': K]\sum_x (D,E)_x \log |k(x)| = \sum_w (\sum_k n_k D_k, \pi^*E)_w \log |k(w)|$$

Therefore

$$[K': K][D,E] = [\sum_k n_k D_k, \pi^*E].$$

It remains to show that

$$[F, \pi^*E] = 0.$$

Since C is projective we may, b^ the Chinese Remainder theorem, find a rational function f on C such that

$$\pi^{-1}(\mathrm{Support}\ (E+(f))) \cap \mathrm{Support}\ F = \emptyset.$$

Therefore

$$[F, \pi^*E] = [F, \pi^*(E+(f)_A)] = 0.$$

In this section we apply the results of section 3 to the Lichtenbaum-Shefarevitch pairing (,).

Let 0 be a discrete valuation ring with a finite residue field. Let p be the maximal ideal of O, K the fraction field. Let C be a regular curve over 0. Let K' be a finite unramified extension of K, O' the integral closure of 0 in K', q the maximal ideal ofj O'. Since O' is etale over O, $C \otimes O'$ is a regular curve over O'.

Let C' be a regular curve over O' obtained by a finite sequence of blow-ups of $C \otimes O'$. Let $\pi: C' \to C$ be the projection. Let D and E be Weil divisors of C, one of which lies over p.

Proposition 11 : $(D,E) = (\pi^*D, \pi^*E)$.

Proof : We may assume that D and E are prime divisors which meet properly and D lies over p. Then

$$(D,E) = \sum_x (D,E)_x \dim_{k(p)} k(x),$$

where the summation runs over points x common to both D and E. Let

$$\pi^*D = \sum n_i D_i + F,$$

as in section 3. By Proposition 9 we have:

$$[K':K](D,E) = \sum_x (\sum_{y \to x} (\sum_i n_i D_i, \pi^*E)_y \dim_{k(x)} k(y)) \dim_{k(p)} k(x)$$

where the first summation runs over points x in C common to both D and E; and for a given x, the second summation runs over points y of C' which map to x. Therefore

$$[K':K](D,E) = \sum_x \Big(\sum_{y \to x} (\sum_i n_i D_i, \pi^* E)_y \dim_{k(p)} k(y) \Big)$$

$$= \sum_x \Big(\sum_{y \to x} (\sum_i n_i D_i, \pi^* E)_y \dim_{k(q)} k(y) \Big) \dim_{k(p)} k(q)$$

$$= (\sum_i n_i D_i, \pi^* E)(\dim_{k(p)} k(q))$$

$$= (\sum_i n_i D_i, \pi^* E)[K':K].$$

We obtain

$$(D,E) = (\sum_i n_i D_i, \pi^* E)$$

It remains to show that $(F, \pi^* E) = 0$. The proof of this is as in the proof of Proposition 10.

Let R be the strict henselization of 0, so R is a complete discrete valuation ring over), with residue field equal to $\overline{k(p)}$, the algebraic closure of k(p) Furthermore R is unramified over 0, meaning P^R is prime in R. Let $\rho: C \otimes R \to C$ be the projection. Let D and E be Weil divisors of C, one of which lies over p.

Corollary : $(D,E) = (\rho^* D, \rho^* E)$

Proof: Follows easily from Proposition 11.

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