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REVIEW ARTICLE

MOVING MESH METHODS FOR PARTIAL DIFFERENTIAL EQUATION

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Moving Mesh Methods for Partial Differential Equation

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In the previous section, we have outlined the aims and some techniques behind the generation of irregular grids. We now turn our attention to methods which aim to move the mesh in time to solve non-steady differential equations. Whilst retaining the properties (and hence the numerical benefits) of the ideas presented above. We shall make constant reference to the techniques in Section 2.1, so it makes sense to follow the same order of events. starting with the use of the equidistribution principle in deriving moving mesh methods in one dimension.

An early incorporation of the equidistribution idea into a moving mesh method is outlined by Petzold. Here a natural extension of the interleaving numerical solution approach for a stationary, adaptive grid is presented. Since the solution of the problem now develops with time, the equidistribution part of the interleaving solution approach is undertaken at intervals, usually chosen by some predetermined error measure, during the forward integration in time. In other words, at certain times throughout the numerical solution of the equation, the grid is reequidistributed, hence moving the nodes throughout time, the solution on the new grid being found via some interpolation process. In a slight variation on this technique Blom et al used a predictive step, reequidistribute the grid using the prediction and then update the solution on the new grid. The update step is written in a Lagrangian form, involving the movement of the nodes in the redistribution, hence no interpolation step is required. The Blom approach bridges the gap between the static, regridding technique of Petzold and more dynamic traditional moving mesh methods. The major difference between the two is the interpretation of mesh speeds included within the solution procedure. We continue this theme further and explore the various forms of this continuously moving mesh idea.

In contrast to the regridding idea, an early dynamic moving mesh technique was devised by Dor & Drury. Here a separate equation for mesh speeds is developed via a function R to control mesh resolution which acts in the same way as a monitor function (despite no formal mention of equidistribution ideas). A simple relation between the speeds of the points $\dot{x}$ and R is solved in conjunction with the underlying PDE. Other early additional moving mesh equations include the work by Adjerid & Flaherty who used a moving mesh equation within a finite, element framework to equidistribute the local discretisation error within the scheme. Petzold followed the regridding approach with a more dynamic moving mesh method, the idea here being that using transformed pseudo-Lagrangian moving mesh coordinates, mesh speeds can be chosen so as to minimise the movement of the mesh in the transformed variables, so the solution in these coordinates is changing as slowly as possible for an easier numerical solution.

White followed earlier grid generation work in one-dimension by using a moving mesh method based upon the transformation to arc-length type coordinates. Applications of early moving mesh methods include the work by Larrouturou working on a flame propagation problem, a single mesh speed being derived for the entire grid, this velocity chosen to preserve thermal energy in the solution, the entire grid is then moved as a rigid body. For the reader's interest, a review of some of the earlier moving mesh methods in one-dimension can be found in Hawken et al .

We now turn our attention to the work of Huang Ren and Russell. In contrast to the work by Dorfi & Drury the moving mesh equation is derived directly from the equidistribution principle. In several moving mesh partial differential equations (MMPDE's) are derived in this manner, with the aims of the resulting algorithm being simple, easy to program and relatively insensitive to the hoice of user-defined parameters. In all seven of these MMPDE's are constructed using three different approaches, the first two of which are motivated by equidistribution. Using the one-dimensional computational and physical coordinate systems as described in Section 2.1 two quasistatic equidistribution principles (QSEP's), are obtained by differentiating the integral form of the equidistribution principle (2.3) with respect to ξ once and twice respectively.

$$M(x(\xi,t),t)\frac{\partial}{\partial \xi}x(\xi,t)=\int_0^1 M(\tilde{x},t)d\tilde{x} \qquad (2.20)$$

and

$$\frac{\partial}{\partial \xi}\left\{M(x(\xi,t),t)\frac{\partial}{\partial \xi}x(\xi,t)\right\}=0. \qquad (2.21)$$

To introduce node movement into the picture, time differentiation is undertaken. Several mesh movement equations have been produced by, for example Anderson Hindman & Spencer and Ren & Russell the former two papers being early attempts with the transformation between physical and computational space, first in one and later in two dimensions owever some of these earlier forms include time differentiation of the integral quantity

$$\theta(t)=\int_0^1 M(\tilde{x},t)d\tilde{x}.$$

Huang, Ren & Russell state, without supporting argument, that the quantity $\theta(t)$ or ts time derivatives are too complicated to include in actual computation. However, by first differentiating the original equidistribution principle with time and then with, twice we obtain

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right)\right\}=0$$

which can be written as (MMPDE1)

$$\frac{\partial}{\partial \xi}\left(M\frac{\partial \dot{x}}{\partial \xi}\right)+\frac{\partial}{\partial \xi}\left(\frac{\partial M}{\partial \xi}\dot{x}\right)=-\frac{\partial}{\partial \xi}\left(\frac{\partial M}{\partial t}\frac{\partial x}{\partial \xi}\right) \qquad (2.22)$$

so giving a moving mesh equation without reference to $\theta(t)$ In the same paper an alternative set of moving mesh equations, MMPDE's 2-4 are derived by considering (2.21) and requiring that the mesh satisfy the condition at the later time $t+\tau$ (where $0<\tau\ll 1$) instead of at time *t* i.e.

$$\frac{\partial}{\partial \xi}\left\{M(x(\xi,t+\tau),t+\tau)\frac{\partial}{\partial \xi}x(\xi,t+\tau)\right\}=0.$$

This equation is thought to be a strong enough condition to regularize the mesh movement by Huang et al. Substituting the expansions

$$\begin{aligned}\frac{\partial}{\partial \xi}x(\xi,t+\tau) &= \frac{\partial}{\partial \xi}x(\xi,t)+\tau\frac{\partial}{\partial \xi}\dot{x}(\xi,t)+O(\tau^2)\\ u(x(\xi,t+\tau),t+\tau) &= u(x(\xi,t),t)+\tau\dot{x}\frac{\partial}{\partial x}u(x(\xi,t),t)\\ &\quad+\tau\frac{\partial}{\partial t}u(x(\xi,t),t)+O(\tau^2)\end{aligned}$$

into (2.2) and dropping higher order terms gives MMPDE 2 (2.23) which in fact is MMPDE1 with an additional 'correction' term

$$\frac{\partial}{\partial \xi}\left(M\frac{\partial \dot{x}}{\partial \xi}\right)+\frac{\partial}{\partial \xi}\left(\frac{\partial M}{\partial \xi}\dot{x}\right)=-\frac{\partial}{\partial \xi}\left(\frac{\partial M}{\partial t}\frac{\partial x}{\partial \xi}\right)-\frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) \qquad (2.23)$$

The extra term is a measure of how well the current grid is equidistributed and hence MMPDE 2 moves the grid towards an equidistributed state even when M is independent of *t*. For this reason, terms involving $\frac{\partial M}{\partial t}$ are less important for MMPDE 2 than MMPDE 1 and disregarding these terms leads to MMPDE's 3 and respectively, i.e.

$$\frac{\partial^2}{\partial \xi^2}(M\dot{x})=-\frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) \qquad (2.24)$$

and

$$\frac{\partial}{\partial \xi}\left(M\frac{\partial \dot{x}}{\partial \xi}\right)=-\frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right). \qquad (2.25)$$

The remaining MMPDE's (5-7) are devised by considering attraction and repulsion pseudo-forces between nodes. Here the mesh movement is specifically motivated by taking the monitor to be some error measure, so nodes are attracted together when the error is larger than average and repelled when the measure is below average. The error is then expressed as an integral over each cell, W_i, usually taking the form

$$W_i=\int_{x_i}^{x_{i+1}} M(\tilde{x},t)d\tilde{x}.$$

MMPDE's (5-7) stem from this relation and all involve the correction term mentioned above. which seems to be a key term as it can determine the time-scale for the mesh movement and hence can be adapted to suit the problem in hand. Moreover since the correction term can be derived from the equidistribution idea, its inclusion in the latter mesh equations suggests that the error is evenly distributed over the mesh and the equidistribution and attraction/repulsion ideas are therefore thought to be closely related. Huang, Ren & Russell also provide theoretical analysis suggesting that the MMPDE's cannot produce instances where nodes cross paths when the MMPDE is solved exactly, indicating stability of the resulting meshes. The stability analysis follows early work by Flaherty et al. In particular it is noted that for MMPDE 1 the mesh would be stable if the measure

$$L(t) = \max_{0 \leq \xi \leq 1} \frac{M(x(\xi,0),0)}{M(x(\xi,t,0)}$$

were to remain bounded. However for most choices of M, L(*t*) is likely to increase, L_i et al, went on to discuss the stability of such moving mesh systems in greater detail.

The resulting equations (MMPDE's 1-7) have spawned a variety of work in various applications, sometimes with a common modification, that being the spatial smoothing of the monitor function M. Dorfi & Drury and Furzeland et al, came to the conclusions in their early moving mesh work that when using finite, difference schemes to approximate derivative terms, in order to obtain 'reasonable' accuracy the mesh should be, in some sense, smoothed. Verwer et al proved that smoothing the mesh is equivalent to smoothing the monitor function over the grid Motivated by this work, Huang, Ren & Russell use MMPDE's (3-7) with a smoothed monitor function $\tilde{M}$ defined at each node by

$$\tilde{M}_i = \sqrt{\frac{\sum_{k=i-p}^{i+p} (M_k)^2 \left(\frac{\gamma}{1+\gamma}\right)^{|k-i|}}{\sum_{k=i-p}^{i+p} \left(\frac{\gamma}{1+\gamma}\right)^{|k-i|}}}$$

Where γ is a smoothing parameter and p is a non-negative integer referred to as the smoothing index which determines the range of the smoothing. These ideas provide a valuable tool in higher dimensions, since using a locally smoothed monitor function is considerably easier than smoothing the entire mesh separately Moreover it is noted in that MMPDE's 3 & 4 permit a possible extension to multidimensions.

Mackenzie and Stockie et al have both applied the smoothed moving mesh equations to PDEs in one-dimension and later to systems of hyperbolic conservation laws, where monitors were not only smoothed but combined to provide a moving grid on which to simulate the development of several time dependent variables. Mackenzie & Robertsonalso used a mesh equation based upon equidistribution applied to a problem involving a phase change. Here a monitor based upon the asymptotic behaviour of the problem was used, clustering nodes around the moving interface, whilst the inclusion of a constant term also allowed sucient nodes to be placed away from the region. Further applications of the MMPDE's (1-7) include work by Qiu & Sloan who applied MMPDE 6 with the outlined technique of smoothing the monitor to Fisher's Equation. Interestingly enough, a new monitor was derived for specific use with reaction- diffusion problems (2.26) after arc-length and curvature monitors proved to be unsuccessful, this was

$$M(x,t) = \left[1 + \alpha^2(1-u)^2 + \beta^2(a-u)^2\left(\frac{\partial^2 u}{\partial x^2}\right)^2\right]^{\frac{1}{2}} \qquad (2.26)$$

where α, β and *a* are user defined parameters.

Huang and Russell also investigated the addition of artificial diffusion terms to the monitor as a means of smoothing, the resulting method satisfying a mesh crossing condition and allowing for possible extension to higher spatial dimensions.

A so-called Moving Mesh DifferentialAlgebraic Equation (MMDAE) is developed by Mulholland, Qiu & Sloan Instead of using the an MMPDE, the mesh movement is prescribed by a QSEP (2.20 & 2.21) written in terms of an algebraic equation involving the stationary grid points and the monitor function M. In fact the algebraic relation is the equidistribution relation written previously in Section 2.1 equation (2.6) This is coupled with the moving grid Lagrangian form of the underlying PDE and integrated forward in time using a first-order backward Euler method (used since these systems tend to be stiff). In this technique is used in conjunction with a pseudo-spectral processing of the solution of hyperbolic problems, Qiu & Sloan continue the work, comparing the method and in particular the stability with the established MMPDE 5 of Huang et al. Of particular interest is the stability of the discrete solution of the steadystate solution to Burgers, equation by examining possible steady solutions arising from the two adaptive discretisations of the unsteady problem.

We now move on to moving mesh methods in higher dimensions. In the previous section we outlined a class of stationary grid adaption methods based upon minimising a mesh generation functional. As with the moving-mesh techniques in one dimension, we introduce mesh speeds into such a grid adaption method so as to preserve the properties of the grid as it moves in time. A popular way to introduce mesh speeds into the mesh functional approach is by use of the so, called gradient flow equations. Following the approach of Huang & Russell a functional $I[\xi, \eta]$ is minimised over the computational domain Ω_c. One way to minimise I is to follow the steepest descent direction given by the first derivative of I. The following 'gradient flow' equations define a flow which converge to the equilibrium state at $t \to \infty$,

$$\frac{\partial \xi}{\partial t} = -\frac{\partial I}{\partial \xi}, \qquad \frac{\partial \eta}{\partial t} = -\frac{\partial I}{\partial \eta}.$$

In practice a modied version of these equations is used in with the inclusion f the familiar correction term τ and the introduction of *P*, an operator on the underlying function space.

$$\frac{\partial \xi}{\partial t} = -\frac{P}{\tau}\frac{\partial I}{\partial \xi}, \qquad \frac{\partial \eta}{\partial t} = -\frac{P}{\tau}\frac{\partial I}{\partial \eta}.$$

The extra term P is used to choose more suitable directions than that of steepest descent with the terms allowing the user to choose a suitable time scale for the problem. It has already been noted in Section 2.1 that the functional approach in one-dimension can be shown to be equivalent to the equidistribution principle. Moreover the approach here can be shown to be similar to using MMPDE 5 being based on the attracting and repellent forces of the monitor function. Indeed Beckett et al used a similar version of the monitor outlined previously (2.26) in conjunction with a onedimensional analogue of (2.29) for the solution of Burgers' equation. More recently MMPDE 5 has been used in two dimensions as part of an adaptive finite element method by Cao et al for the solution of a combustion problem consisting of coupled nonlinear reaction-diffusion equations.

Huang & Russell give multi-dimensional generalisations using this methodology for MMPDE's 4 and 6 Using this approach and the general grid generation functional (2.16), a suitable P is given in terms of the determinants of the two monitor matrices, i.e. $\tilde{g_1} = det(G_1)$ and $\tilde{g_2} = det(G_2)$, giving the resulting MMPDE

$$\frac{\partial \xi}{\partial t} = -\frac{1}{\tau\sqrt{\tilde{g_1}}}\frac{\partial I}{\partial \xi}, \qquad \frac{\partial \eta}{\partial t} = -\frac{1}{\tau\sqrt{\tilde{g_2}}}\frac{\partial I}{\partial \eta} \qquad (2.27)$$

or

$$\frac{\partial \xi}{\partial t} = -\frac{1}{\tau\sqrt{\tilde{g_1}}}\nabla.(G_1^{-1}\nabla\xi), \qquad \frac{\partial \eta}{\partial t} = -\frac{1}{\tau\sqrt{\tilde{g_2}}}\nabla.(G_1^{-1}\nabla\eta). \qquad (2.28)$$

As with solving for a stationary mesh, the actual computations are carried out after interchanging dependent and independent variables, giving

$$\begin{aligned}\frac{\partial \overline{x}}{\partial t} = & -\frac{\overline{x}_\xi}{\tau\sqrt{\tilde{g}_1}J}\left\{+\frac{\partial}{\partial \xi}\left[\frac{1}{Jg_1}(\overline{x}_\eta^T G_1 \overline{x}_\eta)\right] - \frac{\partial}{\partial \eta}\left[\frac{1}{Jg_1}(\overline{x}_\xi^T G_1 \overline{x}_\eta)\right]\right\} \\ & -\frac{\overline{x}_\eta}{\tau\sqrt{\tilde{g}_2}J}\left\{-\frac{\partial}{\partial \xi}\left[\frac{1}{Jg_2}(\overline{x}_\eta^T G_2 \overline{x}_\eta)\right] + \frac{\partial}{\partial \eta}\left[\frac{1}{Jg_2}(\overline{x}_\xi^T G_2 \overline{x}_\xi)\right]\right\} \qquad (2.29)\end{aligned}$$

where J is the Jacobian of the coordinate transform.

Given this general framework, equivalentMMPDE's can be constructed using the various specific functionals described in Section 2.1 Dirichlet boundary conditions are preferred for the solution of (2.29) as this yields a unique solution, but for many problems this is not applicable since the boundary may not be stationary. Indeed, in some cases it is useful to moves nodes around the fixed boundary, for which many techniques are under investigation, the most popular being preserving a onedimensional arc-length equidistribution of nodes on the boundary (see Huang & Russell, Beckett et al).

Huang and Russell, outline a familiar interleaving approach for the solution of the higher dimensional MMPDE combined with the underlying physical PDE as follows :

- Calculate the monitor functions G_1 and G_2 on the current mesh.
- Update the mesh at time $t + \Delta t$ by integrating the MMMPDE(2.29) keeping G_1 and G_2 constant.
- Integrate the physical PDE to get the solution at time $t + \Delta t$ using the mesh

$$\underline{x}(t) = \underline{x}^n + \frac{(t - t_n)}{\Delta t_n}(\underline{x}^{n+1} - \underline{x}^n)$$

and mesh speed

$$\dot{x}(t) = \frac{(x^{n+1} - x^{n})}{\Delta t}.$$

- Choose a value of Δt_{n+1} for the next time step from the physical PDE.

As with their work in one-dimension. Huang, Ren & Russell suggest that the time correction term is preset by the user or determined by the development of the solution. However the choice of this value in one-dimension is relatively insen, sitive and it is thought to be so in higher dimensions also. Central finite difference discretisations are used by Huang & Russell along with a simple rectangular uniform reference mesh for the computational space. Again, extending the work carried out in one-dimension, the monitor is smoothed locally.

On reflection, the functional framework for multidimensional moving mesh methods gathers together all of the work described, both in grid adaption and one-dimensional moving grid techniques, since the strict equidistribution ideas in one-dimension can be written in terms of a functional and the moving mesh methods in higher dimensions are derived from a functional approach to grid adaption.

As an interesting aside, work by Demirdžić & Perić considers moving mesh methods from a more practical aspect. The authors suggest that many of the moving mesh algorithms before them induce error by not satisfying exactly any relevant conservation laws. Work is continued mesh movement equations are derived for the solution of the Navier Stokes equations from a general scalar quantity conservation law. The fact that relevant physical quantities are conserved almost 'by construction' in the method is considered to be of utmost importance and is the driving force behind the moving grid.

Whichever approach is undertaken, a good understanding of the numerical techniques alone may not be good enough for the solution of some problems. We shall continue in the next section by introducing recent work which combines moving mesh methods and self-similar solution techniques, which suggests reasonable choices of monitor functions for certain problems. In particular we shall consider application to the solution of the PME, which we now describe in detail.

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