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REVIEW ARTICLE

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Spherical Codes, Cubature Formulas and Designs

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INTRODUCTION

A spherical code is a nonempty finite subset X C S(E). Its angle set is defined as

$$\mathbf{A(X)} = \{\langle \mathbf{x}, \mathbf{y} \rangle : \mathbf{x}, \mathbf{y} \S \mathbf{X}, \quad \mathbf{x} \neq \mathbf{y}\} \mathrm{C}[-1, 1)$$

In terms of the Coding Theory the set across A(X) is the distance set. The following result is purely combinatoric because X is arbitrary. Lemma is the main key to it.

THEOREM **.** Let X be a spherical code, |X | = n, |A(X)| = s. Then

$$n <= \binom{m+s-2}{m-1} + \binom{m+s-1}{m-1}$$

Proof. Consider the polynomial F; deg F = s; such that F|A(X) = 0. Then F(1) ≠ 0 since

1 €A(X): Now we apply Lemma . The identity turns into

$F(1) \sum | \lambda(x)|^2 = \sum c_{m,k} (F) \sum |u_{ik}(x)|^2$

where

$u_{ik} = \sum S_{ki} (x) \lambda(x)$

are linear forms of the vectors [λ (x)]x ∈ X: The left expression in shows that the rank of the Hermitian quadratic form is equal to n. The right expression shows that the same rank does not exceed the total number of summands there,

$n <= \sum h_{m,k}$

k=0

n < = dim Pol(E)

Now we pass to the special spherical codes which are so nicely arranged on the sphere that the integration over the sphere is equivalent to a weighted averaging over a code. Certainly, this is possible only for finite-dimensional spaces of integrands, say, for Pol(E; d); Pold(E), etc., but with an upper bound for d.

DEFINITION : A spherical cubature formula of index d is an identity

$\int Ødq = \int Ø \, d\sigma$, Ø ∈ Pol (E;d)

where **σ** is the normalized Lebesgue measure on the sphere S(E) and q is a normalized finitely supported measure,

suppq = {xk}1 ⊂ S(E)

$\int Ødq = \int_{suppq} Ø \, dq = \sum Ø(x) q(x) = \sum_x Ø(x) q(x)$

The points x1; : : : ; xn are called the nodes and its measures qk = q(xk); 1<= k<= n; are called the weights. The set suppq is called the support of the spherical cubature formula.

REMARK : Formula with even d automatically implies that q is normalized since 1 Pol(E; d) in this case.

The identity can be also rewritten as

$\sum Ø(x_k) q_k = \int Ø \, d\sigma$, Ø∈ Pol (E;d)

where qk > 0; 1<= k<= n:

A spherical cubature formula of index d in the case of equal weights, i.e with

$Q_1 = qn = 1/n$

is called a Chebyshev type cubature formula. Its support is called a spherical design of index d, a term from the Algebraic Combinatorics. Thus, the left integral is equal to the arithmetic mean of Á over any spherical design X of index d,

1/ |X| ∑ Ø(x) = ∫ Ø dσ , Ø ∈ Pol(E;d)

In the Algebraic Combinatorics the spherical cubature formulas are also called the weighted spherical designs.

PROPOSITION : A spherical code X is a spherical design of even index d if and only if

∑ Ø(gx) = ∑ Ø(x) , Ø ∈ Pol(E;d) , For all , g ∈ O(E) **.**

Proof. This immediately follows from the unitary invariance of the measure **σ**. The identity means that the linear functional

Ø → 1/|X| ∑ Ø(x)

is orthogonally invariant. On the other hand, this functional can be represented in the Riesz form,

Ø → ∫ ψ Ø dσ

where ψ∈ Pol(E;d) and ψ is uniquely determined. Hence, the function ψ must be orthogonally invariant. Since the action of the orthogonal group on S(E) is transitive, ψ(x) = const.

Note that if d is odd (i.e. ∈d = 1) then

∫ Ø dσ = 0 , Ø ∈ Pol(E;d)

∫ Ø dq = ∑ Ø(x) q(x) = 0, Ø ∈ Pol(E;d)

A spherical code X is called antipodal if X = -X, i.e. with every point x∈ X the opposite

point -x also belongs to X. Obviously, any antipodal spherical code is a spherical design of index d for any odd d. However, it is easy to construct a spherical design of index 1 which is not antipodal. Indeed, for d = 1 is equivalent to

$$\sum_{x \in X} x = 0$$

which just means that the baricenter of the set X is at the origin. Obviously, it includes not only antipodal spherical codes. A spherical code X is called podal if X \ (-X) = Ø;, i.e. for every x ∈ X its opposite point -x does not belong to X.

PROPOSITION : A spherical cubature formula of index d is equivalent to the system of equalities

∫ Ø dq = ∑ Ø(x) q(x) = 0, Ø ∈ Harm(E;k)

And, in addition

∫ dq = ∑ q(x) = 1

in the case of even d .

Proof. The cubature formula implies since Ø ∈ Harm(E; k) so, Ø ∈ Pol(E; d) for k ∈ Ed according . Thus, it is applicable R to Ø 2 Harm(E; k); k 2 Ed. It remains to note that Ø d¾ = 0 by (3.60). In addition, if d is even.

Conversely, the harmonic decomposition

Ø = ∑ $Ø_k$, yields

∫ Ø dσ = ∑ ∫ $Ø_k$ dσ = ∫ $Ø_{ed}$ dσ

The integral is Ø0 if ∈d = 0 since Ø 0 = const and ¾ is normalized. If ∈d = 1 then the integral is equal to zero.

Similarly,

∫ Ø dq = ∫ $Ø_{ed}$ dq

COROLLARY : A spherical code X is a spherical design of index d if and only if

∑ Ø (x) =0 , Ø ∈ Harm(E;k) , k∈ ∈d , k>=1.

COROLLARY : A spherical cubature formula of index d is a spherical cubature formula of every index k ∈ Ed:

DEFINITION : A spherical cubature formula of degree d (or strength d) is a spherical

cubature formula of all indices 2t; 2t - 2; : : : ; 0; i.e.

∫ Ø dq = ∫ Ø dσ , Ø∈ Pol(E;k) , 0<=k<=d

In view of the homogeneous lifting this definition is equivalent to

∫ Ø dq = ∫ Ø dσ , Ø∈ Pol(E)

PROPOSITION : A spherical cubature formula of degree d is the same as a spherical cubature formula of indices k = d - 1; d.

COROLLARY : Any antipodal spherical cubature formula of even index d has also degree d.

COROLLARY : For any spherical cubature formula of even index d its symmetrization

X → X` = X U (-X) , Q → Q` , Q`(+-x) = ½ Q(X)

yields an antipodal spherical cubature formula of degree d.

In particular, the symmetrization generates a correspondence between podal and antipodal spherical codes such that each antipodal spherical code can be obtained from exactly $2^{|X|}$ podal codes.

PROPOSITION : A spherical cubature formula of degree d is equivalent to the system of equalities.

∫ Ø dq = ∑ Ø(x) q(x) = 0, Ø ∈ Harm(E;k) , 1 <= k <= d.

COROLLARY : A spherical code X is a spherical design of degree d if and only if

∑ Ø(x) = 0 , Ø ∈ Harm(E;k) , 1 <= k <= d.

A spherical design of degree d is called a d-design. Now let us note that the upper bound is valid for any cubature formula as a spherical code with special properties. Below we establish a lower bound for the number

n= |X| = |suppq|

of nodes of a spherical cubature formula of even index d. If d is odd then there is no nontrivial lower bound for n. Indeed, any pair of mutually opposite points on the sphere is a spherical design of index d.

THEOREM : For any spherical cubature formula of even index d the inequality holds.

n >= (m + s – 2)

(m – 1)

n >= dim Pol(E;d/2)

Proof. Suppose to the contrary and then consider the interpolation problem for θ ∈ Pol(E; d/2):

θ (xk) = 0, 1 <= k <= n

This problem has a nontrivial solution. By the given cubature formula for the function Ø= | θ | ^{2} ∈ Pol(E; d/2): we get

∫ | θ | ^{2} dσ = 0

whence θ = 0, the contradiction.

The problem arises whether the lower bound is exact and, in the case of affirmative answer, what is the corresponding cubature formula.

DEFINITION : A spherical cubature formula of even index d is called tight if in the equality is attained.

It is easy to see that the support X of any tight spherical cubature formula is podal. Indeed, if x ∈ X \ (-X) then one of nodes x0 or -x0 can be omitted since Ø (-x0) = Ø (x0): Thus, the number of nodes becomes less than lower bound .A spherical code X = (xk)n is called t-interpolating if for every vector [³k]n there exists a unique form Ø ∈ Pol(E; t) .

LEMMA : A spherical code X = (xk)n is t-interpolating if and only if

n = (m + t – 1)

(m – 1)

and there is no a nontrivial form Ø ∈ Pol(E; t) such that Ø |X = 0.

THEOREM . If a spherical cubature formula of index d = 2t is tight then

(i) its support (xk)n $_1$ is a t-interpolating system;

(ii) the corresponding Lagrange basis (Lj)n $_1$ is orthogonal, i.e.

(Lj , Lk) = ∫ Lj` Lk dσ = 0 , j ≠ k.

(iii) the weights are

Qj = || Lj||^2 = ∫ |Lj|^2 dσ , 1 <= j <= n.

Conversely, let a spherical code (xk)n $_1$ be a t-interpolating system with property (ii): Then (xk)n $_1$ is the support of the tight spherical cubature formula of index d = 2t with weights .

Proof. Let the formula be tight. Then for any form Ø ∈ Pol(E; t) ,we have

∫ | Ø |^2 dσ = ∑ | Ø (xk) |^2 qk.

If Ø (xk) = 0; 1 <= k<=· n; then Ø = 0, i.e. the mapping Ø → (Ø (xk))n $_1$ is injective. Moreover, n = dim Pol(E; t). As a result, we get (i) by Lemma. Using the cubature formula with Ø = Lj` Lk or jLkj^2 we get (ii) and (iii) . Conversely, any form Ø ∈ Pol(E; d) is a linear combination of the products LjLk ∈ Pol(E; d),

Ø = ∑ b_{jk} Lj` L_k

with some coefficients bjk. (Indeed, any monomial of degree d is product of two monomials of degree t and the latter can be both decomposed for a basis (Lj):) Hence,

$\sum Ø(x1)\ qt = \sum b_{jk}`\ Lj`(x1)`\ q1 = \sum b_{kk}.\ Q_k$

So, the cubature formula under consideration is of index d. The formula is tight by Lemma Below we need the following statement of a general nature.

LEMMA : For a fixed pole x∈ S(E) the evaluation functional Ø → Ø (x) on Pol(E; t) can be represented as

Ø(x) = ∫ θx` (y) . Ø (y) dσ (y)

where θy ∈ Pol(E; t): The norm || θx || does not depend on y. Therefore, || θx || only depends on m and t.

Proof. The first statement is actually the Riesz representation of the evaluation functional. Further, implies

∫ θgx` (y) . Ø (y) dσ (y) = ∫ θx`(y) . Ø (gy) dσ (y) , g ∈ O(E).

Taking Ø (y) = θgx(y) we obtain,

$|| θgx ||^2 <= || θx ||^2 \int | θgx (gy) |2\ dσ (y)$

by the Schwartz inequality. Since the measure ¾ is orthogonally invariant, the latter integral is equal to kµgxk so, the inequality takes the form kµgxk · kµxk for all y 2 S(E) and all g ∈ O(E). Changing g for g¡1 and x for gx we obtain the converse inequality. Thus, kµgxk = kµxk and it remains to recall that O(E) acts on S(E) transitively. Combining this lemma with Theorem we obtain,

COROLLARY : In any tight spherical cubature formula the weights are equal. In other words, the support of any tight spherical cubature formula of index d is a spherical design of index d (a tight spherical design). As we already know, this spherical design is podal.

Proof. Since the basis (Lj) ½ Pol(E; t) is orthogonal, we have the decomposition

$θx = \sum (θx , Lj) / || Lj ||^2 . Lj$

Whence

θx (y) = ∑ Lj (x). Lj (y) / qj

In particular,

θxk (y) = Lk(y) / qk , 1<= k <= n ,

since (Lk) is the Lagrange basis. Passing to the norms ,we obtain

|| θxk || = || Lk || / qk = 1/ √qk

It remains to apply Lemma again.

REMARK : As follows from the proof

$|| θx ||^2 = n = (m + t – 1)$

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