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REVIEW ARTICLE

MATHEMATICAL CONCEPTS AND SOCIETIES OF AGENTS

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Mathematical Concepts and Societies of Agents

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INTRODUCTION

Historians and Sociologists such as Latour (1987) have critiqued the traditional scientific method and emphasized that science can only be understood through its practice. Feminist critics such as Gilligan (1986) and Belenky et al (1986) have criticized rationalism from a psychological perspective. They show how the formal knowledge of rationalist procedures has created a “separate knowing” which has alienated many and women in particular. They propose a revaluing of personal knowledge and what they call “connected knowing” - a knowing in which the knower is an intimate part of the known.

In his book “The Society of Mind”, Marvin Minsky (1989) describes the workings of the human mind in a radically decentered way. We are, says Minsky, a collection of thousands of distributed “agents”, each by itself stupid, specialized, able to communicate only a very small part of what it knows to any other agent.

One example which Minsky and Papert analyze through the society of mind theory is the water volume experiment of Piaget (1954). In this classic experiment, a child is shown two full water glasses, one taller and one wider glass. He/she is then asked which glass has more water. Children before a certain stage (usually around age 7) say that the tall glass has more. If the tall glass is then poured into another wider glass and fills it, then if the children are asked which of the two full glasses has more, they say the two glasses have the same amount of water. But then if the water is poured back into the tall glass, they again assert that the tall glass contains more water.

This phenomenon is robust and has resisted all attempts to explain it away as merely a semantic artifact. Piaget explains the children’s behavior as a construction of the concept of liquid volume. Before roughly age 7, children do not conserve volume, liquids can become more or less if their appearance is altered.

Minsky and Papert’s analysis of this situation posits the existence of an MORE Agency (a system or society of agents) and three sub-agents:

TALLER: This agent asserts which of two glasses is taller.

WIDER: This agent asserts which of two glasses is wider.

HISTORY: This agent asserts that two glasses have the same amount of liquid if nothing has been added or removed.

In the pre-conservational state, a child’s MORE agency can be thought of as connected as follows:

MORE TALLER WIDER HISTORY

If TALLER asserts itself, then the MORE agency listens to it. If TALLER is quiet, then MORE listens to WIDER. If WIDER is also silent, then MORE listens to HISTORY. Thus, the child’s agents are arranged in a linear order. If one glass is taller than the other, MORE will assert that that glass has more liquid.

But post-conservation, say Minsky and Papert, the child’s agents are connected differently. An APPEARANCE agent is created that subsumes TALLER and WIDER.

The APPEARANCE agent functions as a middle manager between MORE and TALLER/WIDER.

In this new organization, MORE listens to APPEARANCE first. But APPEARANCE listens to both TALLER and WIDER. If only one of these asserts itself or if they agree, then APPEARANCE speaks up about which glass has more. However, if the two agents TALLER and WIDER disagree, then the “principle of non-compromise” is invoked and APPEARANCE is silent. MORE, then, listens to HISTORY in making its decision.

MORE APPEARANCE HISTORY TALLER WIDER

Minsky summarizes the lessons of this example as Papert’s principle:

“Some of the most crucial steps in mental growth are based not simply on acquiring new skills, but on acquiring new administrative ways to use what one already knows.”

An important corollary of Papert’s principle is that, when new skills are developed, it doesn’t follow that old habits, intuitions, agents are dispelled. They may still resurface in other contexts. But their connections in a particular agency will have been greatly altered.

While it happened here that the few agents in this example are organized hierarchically, this can be expected to be true in only the simplest cases. Nor is it generally true that all agencies have a clear “execution model”. In the general case, agencies interpenetrate, call each other’s' sub-agents and super-agents and form a tangle of connections. As Minsky says:

“In its evolutionary course of making available so many potential connections, the human brain has actually gone so far that the major portion of its substance is no longer in its agencies but constitutes the enormous bundle of nerve fibers that potentially connect those agencies. The brain of Homo Sapiens is mainly composed of cabling.”

One way then to measure the strength of an agent or agency is to see how well connected it is. We can then recharacterize our definition of concrete in agent language: an agent is more or less concrete depending on the number of agents to which it connects.

The water volume experiment also illustrates the value of agent conflict as an impetus to learning. It was the conflict between the TALLER and the WIDER agents that led to the silence of the APPEARANCE agent and the subsequent attention to

HISTORY. In order for this conflict to be detected, TALLER and WIDER had to both be relatively strong. If TALLER continued to dominate WIDER as it did in the first figure then the appearance of APPEARANCE would do no good. Thus, it was the conflict between two strong trusted agents that motivated the reordering and enrichment of the child’s MORE agency.

In the course of this thesis, we shall draw a strong analogy between intellectual development in adults and conservation experiments of Piaget. The child moves from agent dominance to agent conflict to a new agent network. This process takes a long time, but results in a new set of intuitions about volume. We take this as a model for adult development as well.

The usefulness of agent conflict in this example suggests that paradox is generally a powerful motivator toward new understanding. In a paradox, two trusted and reliable arguments (or agents) that have done their jobs faithfully before with good results are seen to be in conflict.

PARADOX

Throughout history, it has often been the case that paradoxes have led to major reconstructions of mathematical and scientific systems. This should no longer surprise us since much of what is paradoxical about paradoxes is their counterintuitiveness. But it is precisely through encountering paradox that intuitions are built. When two trusted agents conflict, we are ripe for new agent links and middle managers which embody new intuitions. Paradox has played this role in the history of science, as Anatol Rapoport (1967) notes:

Paradoxes have played a dramatic role in intellectual history, often foreshadowing revolutionary developments in science, mathematics and logic.

Whenever, in any discipline, we discover a problem that cannot be solved within the conceptual framework that supposedly should apply, we experience shock.

The shock may compel us to discard the old framework and adopt a new one. It is to this process of intellectual molting that we owe the birth of many of the ideas in mathematics and science....Zeno’s paradox of Achilles and the tortoise gave birth to the idea of convergent infinite series. Antinomies (internal contradictions in mathematical logic) eventually blossomed into Gödel’s theorem. The paradoxical result of the Michelson-Morley experiment on the speed of light set the stage for the theory of relativity. The discovery of the wave-particle duality of light forced a reexamination of deterministic causality, the very foundation of scientific philosophy, and led to quantum mechanics. The paradox of Maxwell’s demon which Leo Szilard first found a way to resolve in 1929, gave impetus more recently to the profound insight that the seemingly disparate concepts of information and entropy are intimately linked to each other.

Rapoport’s examples are among the most salient, but many more mundane examples are to be found in the history of mathematics and science. Questions such as the ones below were hotly debated paradoxes and perplexed mathematicians in their times.

• What is the result of subtracting a larger number from a smaller?

• What is the result of dividing a positive number by a negative number?

• What is the square root of a negative number?

• Which are there more of: integers or even integers?

• How can a finite area have infinite perimeter?

• How can an quantity be infinitely small, yet larger than zero?

But parallel to their phylogenetic importance, the analogy of paradox to agent conflict lets us see their important role in ontogenetic development. It is through the resolution of these conflicts that new meanings and epistemologies are negotiated. But to be of value in development, both agents in a paradox must be strong and trustable. If one is much more trustable than the other then, like TALLER, it will dominate and no paradox will even be perceived. If both agents are weak, then the paradox is not compelling and only serves to undermine further the credibility of each agent.

In order to foster development then, we need to strengthen each of the conflicting agents. If, as is so often done in mathematics classes, we resolve the conflict prematurely by declaring one agent a victor, as being the “right answer”, the “right way to do it”, or the “right definition”, then we undermine the processes that work to concretize these agents, linking them in to the network of agents. Unless we experience the conflict, see which way we might have gone if we hadn’t gone the “right” way, then we will get lost on the next trip.

The result of only getting the “right answers” is brittle formal understanding. We shall see in the probability interviews in Chapter VIII examples of all of the above responses to paradox.

CONCEPTS ARE MESSY

Mathematicians have argued that proofs are the essence of their enterprise. What distinguishes Mathematics from other disciplines is the certainty that is obtained through the rigor of proofs. But in fact proofs are not the source of mathematical certainty. They are a technique used by mathematicians to create a uniform procedure for verification of mathematical knowledge. The technique consists of “linearizing” the complex structure that constitutes the ideas in a mathematical argument. By means of this linearization, mathematical proofs can be checked line by line, each line either an axiom or derived from previous lines by accepted rules of inference.

But the hegemony of the standard style of communicating mathematics (definition/theorem/proof) constitutes a failure to come to terms with the mind of the learner. We attempt to teach formal logical structures in isolation from the experiences that can connect those structures to familiar ideas. The result is that the idea too often remains “abstract” in the mind of the student, disconnected, alien, and separate, a pariah in the society of agents.

Even if the motivation is there to communicate our mathematical ideas to those who don't already understand them, we may have lost the organization of mental agents that existed before the idea was acquired, and forgotten the mechanisms by which we ourselves acquired the ideas. This is clearly illustrated by the conservation experiments of Piaget (1952). In a typical such experiment, a child is shown a tall thin glass filled with water which is then poured into a shorter wider glass. When asked which glass contains more water, so-called “pre-conservation” children say that the tall glass has more. A year or so later, these same children now “conservational” are shown video tapes of their earlier interviews on the subject. These children do not believe that they could ever have made such a “ridiculous” claim. “Of course the glasses contain the same amount of water- the videotapes must have been faked.” (Papert, 1990). Piaget's experiments are dramatic examples of what is a quite common phenomenon -- people have a hard time remembering what it was like to not understand a concept which they now understand.

What is it like to not accept the identity of objects through time? What is it like to not understand what addition is? Because of this phenomenon, a well-meaning mathematical educator often cannot reconstruct the process by which he/she was convinced of the efficacy of a definition or the validity of a proof and mistakenly believes that the linear proof contains all the information and structure necessary for the conceptual understanding.

If asked to justify the formal style of their expostulation, an author or teacher of mathematics may respond that the linear structures (definition/theorem/proof) capture most economically the essence of the material. But it may be that this reason for hiding the messy structure of mathematical ideas is not the whole story. Revealing that the structure of mathematics in your head does not mirror the clean elegant lines of the mathematical text can be quite embarrassing. Yet to reveal that process may be perceived as an admission of vulnerability and weakness. The theorem and proof is the logical mental construct of which we are proud; the web of connections, intuitions, partial understandings, and mistakes from which that logical construct arose may be a source of shame. The traditional form of mathematical expostulation is a shield that mathematical culture has developed to protect the mathematical teacher or author from embarrassment.

It allows the mathematician to present a controlled and logically inexorable understanding without exposing him to the risk of revealing his own messy internal thought processes.

COVERING UP IS DAMAGING

This covering up of the hidden messy structure of mathematical ideas can be damaging to the mathematical learner. If learners believe that the mathematics as presented is a true picture of the way the mathematics is actually discovered and understood, they can be quite discouraged. Why is their thinking so messy when others' is so clean and elegant? They conclude that clearly mathematics must be made only for those born to it and not available to mere mortals. Mathematical discourse is not a form of persuasion continuous with daily discourse, but is instead in some special province all its own, a purely formal phenomenon. These mathematical learners are deprived of the experience of struggling for a good definition, and the knowledge that mathematical truths are arrived at by a process of successive refinement not in a linear and logically inexorable fashion. (see Lakatos, 1976)

Unfortunately, this kind of mathematical culture is self-reinforcing. Those who survive the stark, formal modes of presentation and manage to concretize mathematical structures sufficiently to pursue a career in mathematics have learned along the way that to reveal their internal thought processes is to violate a powerful social norm. In parallel to the standard mathematical curriculum, the student has learned the following “lessons” of mathematical culture:

• No one is interested in your personal constructions of mathematical ideas.

• There is one canonical way of understanding mathematical objects.

• The correct procedure for understanding mathematical ideas is to replace your former unrigorous intuitions with the received rigorous definitions.

• Mathematics is to be grasped instantaneously - struggle with an idea is a sign of dullness and lack of ability.

The student in a mathematics classroom who braves a revelation of his/her tentative understanding of an idea is too often confronted with a demeaning response, “How can you not see the answer to that? -- it’s trivial.”

Not only does this culture impede learning, it is almost certainly an inhibition to new mathematical discovery. In fact, embarrassment at expressing new, half-formulated ideas is a powerful force for conservatism in mathematics. It is difficult to challenge old ideas, or to formulate new ones, in the absence of a culture that supports the floundering, messy process of mathematical exploration.

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