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Principle of Riemann Zeta Functions and Their Strategies

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Abstract – Zeta or L - functions are modelled on the Riemann's zeta function originally defined by the series $\zeta(s) = \sum_{n\geq 1} n^{-s}$ and then extended to the whole complex plane. The zeta function has an "Euler product", a "functional equation" and though very much studied still keeps secret many of its properties, the greatest mystery being the so-called Hiemann Hypothesis. Many similar (or thought to be similar) series $\sum_{n\geq 1} a_n n^{-s}$ have been introduced in arithmetic, algebraic geometry and even topology, dynamics (we won't discuss the latter). We plan basically to discuss zeta functions attached to algebraic varieties over finite fields and global fields.

The first applications of zeta functions haw been the arithmetic progession theorem (Dirichlet. 1837) "there exists one (hence infinitely) prime congruent to a modulo b. whenever a and b are coprime" and the prime number theorem (Kiemann 1859, with an incomplete proof: Hadamard and de la Vallee Poussin, 1896) "the number of primes less than x is asymptotic to $x/\log x$". But further applications were not restricted to the study of prime numbers, they include the study of the ring of algebraic integers, class field theory, the estimation of the size of solutions of (some) diophantine equations, etc. Moreover L-functions have provided or suggested fundamental links between algebraic varieties (motives over Q). Galois representations, modular or au- tomorphic forms: for example, though they do not appear explicitly in Wiles work, it seems fair to say they played an important role in the theory that finally led to the solution of Shimura-Taniyama-Weil conjecture and thus of Fermat's Last Theorem.

The first four lectures develop results and definitions which though all classical are perhaps not too often gathered together. The first lecture introduces Riemann's zeta function, Dirichlet L-function associated to a character. Dedekind zeta functions and describes some applications of zeta functions: the second introduces the Hasse-Weil zeta functions associated to algebraic varieties defined over a finite field, a number field or a function field as well as L- functions associated to Galois representations and modular forms; the third reviews techniques from complex analysis and estimates for zeta functions: the fourth touches the theory of special values of zeta functions, some known like the class number formula and some conjectured like the Birch and Swinnerton- Dyer formula. The fifth and final lecture is an exposition of recent work of Boris Kunyavskif, Micha Tsfasman. Alexei Zykin. Amflcar Facheco and the author around versions and analogues of the Brauer-Siegel theorem.

Frerequisite will be kept minimal whenever possible : a course in complex variable and algebraic number t heory, a bit of Galois theory plus some exposure to algebraic geometry should suffice.

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INTRODUCTION

Let $\mathbb{F}_q$ be a finite field of ***q*** elements and ***p*** its characteristic. Let .V be an algebraic set defined over $\mathbb{F}_q$. For each positive integer k. let N_k denote the number of $\mathbb{F}_{q^k}$-rational points on *X.* The zeta function *Z(X)* of *X* is the generating function

$$Z(X) = Z(X,T) = \exp\left(\sum_{k=1}^{\infty} \frac{N_k}{k} T^k\right).$$

The zeta function contains important arithmetic and geometric information concerning X. It has been studied extensively in connection with the celebrated Weil conjectures.

Both practical applications and theoretical investigations make a good understanding of the zeta function from an algorithmic point of view increasingly important. The aim of this paper is to present a brief introductory- account of the various fundamental problems and results in the emerging algorithmic theory of zeta functions. We shall focus on general properties rather than on results that are restricted to special **cases.** In particular, in most of this paper we do not assume *X* to be smooth and projective, although in that case one can often say more.

The contents are organized as follows. In Section 2 we review general properties of zeta functions from an algorithmic point of view. A naive effective algorithm for computing the zeta function is given. If the characteristic ***p*** is small, one can use Dwork's p-adic method to obtain a polynomial time algorithm for computing the zeta function in the case that the numbers of variables and defining equations for *X* are fixed.

We show that the general case of algebraic sets can be reduced in various ways to the case that A" is a hyper surface with emphasis on the smooth projective case. We consider the complex pure weight decomposition. Using the LLL factorization algorithm and Deligne's main theorem, we show that, when the zeta function is given, one can compute in polynomial time how many zeros and poles with a given complex absolute value it has which is devoted to the P-adic pure slope decomposition, we use the theory of Newton polygons to obtain a similar result for the number of zeros and poles with a given p-adic absolute value.

An algorithm for the simpler problem of computing the zeta function modulo ***p.*** This algorithm shares several characteristic features with the general P-adic method for computing the full zeta function that is presented in [Lauder and Wan 2008] in this volume. Section 7 may thus serve as an introduction to that article.

All algorithms in this paper are deterministic. Probabilistic algorithms will not be discussed. Time is measured in bit operations.

THE BEHAVIOUR OF

$\zeta(s)$ **near** $s = 1$

Starting from the formula

$$\frac{1}{n^s} = s\int_n^\infty \frac{dt}{t^{s+1}} = s\sum_{k=n}^{\infty}\int_k^{k+1}\frac{dt}{t^{s+1}},$$

a reordering of the summations gives, for $\Re(s) > 1,$

$$\zeta(s) = s\sum_{n\geq 1}\sum_{k\geq n}\int_k^{k+1}\frac{dt}{t^{s+1}} = s\sum_{k\geq 1}\left(\sum_{n\leq k}\int_k^{k+1}\frac{dt}{t^{s+1}}\right) = s\sum_{k\geq 1}k\int_k^{k+1}\frac{dt}{t^{s+1}}.$$

The last summation writes in the form

$$\zeta(s) = s\int_1^\infty \frac{[t]}{t^{s+1}}\,dt = \frac{s}{s-1} - s\int_1^\infty \frac{\{t\}}{t^{s+1}}\,dt, \qquad (1)$$

where $[t]$ denotes the integer part of ***t*** and $\{t\} = t - [t]$ its fractional part. Notice

that the formula (1) is an alternative way to obtain the analytic continuation of $\zeta(s)$ in the half plane $\Re(s) > 0.$

When -s = 1, the last integral in (1) is equal to

$$\int_1^\infty \frac{\{t\}}{t^2}\,dt = \lim_{N\to\infty}\sum_{n=1}^{N}\int_n^{n+1}\frac{t-n}{t^2}\,dt = \lim_{N\to\infty}\int_1^N \frac{dt}{t} - \sum_{n=1}^{N}\frac{1}{n+1} = 1-\gamma,$$

where γ Is the ***Euler constant.***

Finally, formula (1) yields the following asymptotic expansion

$$\zeta(s) = \frac{1}{s-1} + \gamma + o(1), \qquad (s\to 1). \qquad \textbf{(2)}$$

This expansion yields interesting results if one computes the expansion obtained by previous equation

$$\zeta(s) = \frac{\eta(s)}{1-2^{1-s}} = \frac{\eta(1)+(s-1)\eta'(1)}{(s-1)\log(2)-(s-1)^2\log^2(2)/2} + o(1)$$
$$= \frac{\eta(1)}{\log(2)(s-1)} + \left(\frac{\eta'(1)}{\log(2)} + \frac{\eta(1)}{2}\right) + o(1).$$

By comparison with (2), we obtain $\eta(1)/\log(2) = 1$ and $\eta'(1)/\log(2) + \eta(1)/2 = \gamma$. In other words; we have obtained the classical result

$$\eta(1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \log(2)$$

and the relation $\eta'(1) = \log(2)(\gamma - \eta(1)/2)$ yields the beautiful series

$$\sum_{n=1}^{\infty}(-1)^n\frac{\log(n)}{n} = \log(2)\left(\gamma - \frac{\log(2)}{2}\right).$$

GENERALIZED EULER CONSTANTS

The expansion (2) can he continued by writing

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n(s-1)^n.$$

The constants γ_n can be proved to satisfy

$$\gamma_n = \lim_{m\to\infty}\sum_{k=1}^{m}\frac{\log^n k}{k} - \frac{\log^{n+1} m}{n+1}.$$

These formula generalize the Euler constant definition (corresponding to the case ***n*** = 0) and for that reason, the constants γ_n are often called the ***generalized***

Euler constants. They were also called ***Stieljes constants*** as they were studied by Stieljes. General informations about these constants can be found on ***Eric Weissten's world of mathematics*** site.

STIELTJES AND HADAMARD

From the Euler product formula we have

$$\frac{1}{\zeta(z)} = \prod_{n=1}^{\infty} (1 - p_n^{-z})$$

for $\Re e\, z > 1$. It follows that

$$\frac{1}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$$

Since ζ has a pole at 1. $1/\zeta$ has a root at 1. It turns out that the series actually converges for $z = 1$ (van Mangoldt 1897. de la Vallee Poussin 1899).

Thus we obtain Euler's curious formula (1748)

$$0 = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{10} - \frac{1}{11} - \frac{1}{13} + \frac{1}{14} + \frac{1}{15} - \frac{1}{17} \cdots$$

Let M be the step function defined by M(0) = 0. *M* piecewise constant, M has a jump $\mu(n)$ at *n.* and M Is equal to the average of its left and right limits at each jump. Then

$$\frac{1}{\zeta(z)} = \int_0^{\infty} x^{-z}\, dM(x) = z \int_0^{\infty} M(x)\, x^{-z-1}\, dx$$

for $\Re e\, z > 1$.

If M grows less rapidly than x^{α} for some $\alpha > 0$ then the second integral above will converge absolutely for $\Re e\, z > \alpha$. Then, by uniqueness of analytic continuation. We can conclude that ζ has no roots in. In 1912 Littlewood proved the converse and therefore:

Theorem 1. *The Riemann hypothesis is true if and only if for each* $\epsilon > 0$ *we have*

$$\lim_{x \to \infty} M(x)\, x^{-1/2-\epsilon} = 0.$$

In 1885 Stieltjes wrote to Hermite that he had proved that $M(x)x^{-1/2}$ is bounded for large x and that therefore the Riemann hypothesis is true. Stieltjes, however, was unable to recall his proof in later years. Hadamard in 1896 published a paper on the zeta function in which he shows that ζ has no roots on the line $\Re e\, z = 1$. He says that he is publishing his proof only because Stieltjes' proof that there are no zeros in $\Re e\, z > 1/2$ has not yet been published.

It now seems likely that Stieltjes had in fact made an error.

OODS AND ENDS OF EULER RELATION

If we form the Dirichlet product we have

$$\zeta(z)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^z}$$

where d(n) is the number of divisors of n.

We have similar series involving sums of divisors of n or the number of positive integers less than *n* relatively prime to*n.*

CONCLUSION

When studying the distribution of prime numbers Riemann extended Euler's zeta function (defined just for *s* with real part greater than one)

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

to the entire complex plane (*sans* simple pole at *s* = 1). Riemann noted that his zeta function had trivial zeros at -2, -4, -6, ...; that all nontrivial zeros were symmetric about the line Re(*s*) = 1/2; and that the few he calculated were on that line. **The Riemann hypothesis is that all nontrivial zeros are on this line.** Proving the Riemann Hypothesis would allow us to greatly sharpen many number theoretical results. For example, in 1901 von Koch showed that the Riemann hypothesis is equivalent to:

$$\pi(x) = \mathrm{Li}(x) + O(x^{1/2} \log x).$$

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