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An Assessment of Aluthge and Duggal Transformations: A Brief Survey

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Abstract – This paper will be appeared in other journal. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An arbitrary operator T in $\mathcal{L}(\mathcal{H})$ has a unique polar decomposition T = UP, where $P = (T^*T)^{\frac{1}{2}} = |T|$ and U is a partial isometry with initial space the closure of the range of |T| and final space the closure of the range of T. Associated with T there is a related operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, sometimes called the Aluthge transform of T because it was studied in the context that T is a p-hyponormal operator (to be defined below). In this note we derive some spectral connections between an arbitrary $T \in \mathcal{L}(\mathcal{H})$ and its associated Aluthge transform $\tilde{T}$ that enable us, in particular, to generalize an invariant-subspace-theorem of Berger to that context. We will also show that the hyperinvariant subspace problems for hyponormal and p-hyponormal operators are equivalent. The following lemma is completely elementary, but sets forth basic relations between T and $\tilde{T}$ that will be useful throughout the paper.

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INTRODUCTION

Let $\mathcal{H}$ be a separable Hilbert space with $2 \leq dim\mathcal{H} \leq \aleph_0$ and $\mathcal{L}(\mathcal{H})$ be the C*-algebra of all bounded linear operators on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$. Let $T = U|T|$ be the unique polar decomposition of T, where U is a partial isornetry such that $kerU = kerT = ker|T|$ and $|T| = (T^*T)^{1/2}$. Obviously $|T|$ is a positive operator.

Also $\| Tx \| = \| |T|x \|$ for all $x \in \mathcal{H}$, and if $E = U^*U$ then E is the initial projection of U (ie., $E = P_{\mathcal{X}}$ where $\mathcal{X} = (kerU)^{\perp} = (kerT)^{\perp}$) and E is the support of T as well as the support of $|T|$. The following definition is due to Aluthge.

Definition 3.1 (Aluthge transformation). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of T, then

$$\tilde{T} = |T|^{1/2}U|T|^{1/2}$$

is called the Aluthge transformation of T.

Definition 3.2 (Duggal transformation). If $T \in \mathcal{L}(\mathcal{H})$ and $T = U|T|$ is the polar decomposition of T, then

$$\widehat{T} = |T|U$$

is called the Duggal transformation of T.

Definition 3.3 (λ—Aluthge transformation). If $T \in \mathcal{L}(\mathcal{H})$, $T = U|T|$ the polar decomposition of T, and $0 < \lambda < 1$, then $\Delta_\lambda(T) = |T|^\lambda U|T|^{1-\lambda}$ is called the λ—Aluthge transformation of T. When $\lambda = 1/2$, the λ—Aluthge transformation is the Aluthge transformation.

The notion of Aluthge transformation was first studied in relation with the p—hyponormal and log —hyponormal operators. Roughly speaking, the Aluthge transformation of an operator is closer to being normal. Aluthge transformation has received much attention in recent years. One reason is the connection of Aluthge transformation with the invariant subspace problem. Jung, Ko and Pearcy proved in that T has a nontrivial invariant subspace if and only if $\widetilde{T}$ does. Another reason is related with the iterated Aluthge transformation.

Foias, Jung, Ko and Pearcy introduced the the concept of Dug- gal transformations, and proved several analogous results for Aluthge transformations and Duggal transformations. Yamazaki proved that for every $T \in \mathcal{L}(\mathcal{H})$, the sequence of the norms of the Aluthge iterates of T converges to the spectral radius $r(T)$. Denning Wang gave another proof of this result. We started studying Aluthge and Duggal transformations hoping to prove, the analogue of the result of Yamazaki, that the sequence of the norms of the Duggal iterates of T converges to the spectral radius $r(T)$, for every $T \in \mathcal{L}(\mathcal{H})$ Several finite dimensional examples suggested that the result is true for Duggal transformations. We succeeded in proving that the sequence of the norms of the Duggal iterates converges to the spectral radius, for certain classes of operators. Further investigation led to an example of a finite dimensional operator showing that there exist operators such that the sequence of the norms of the Duggal iterates does not converge to the spectral radius.

ITERATED ALUTHGE AND DUGGAL TRANSFORMS

Let $\mathcal{H}$ be a Hilbert space and T a bounded operator defined on $\mathcal{H}$ whose (left) polar decomposition is $T = U|T|$. The Aluthge transform of T is the operator defined by

$$\Delta\,(T) = |T|^{1/2}U\;|T|^{1/2} \quad (3.3)$$

This transform was introduced in by Aluthge, in order to study p-hyponormal and log-hyponormal operators. Roughly speaking, the idea behind the Aluthge transform is to convert an operator into another operator which shares with the first one some spectral properties but it is closer to being a normal operator.

The Aluthge transform has received much attention in recent years. One reason is its connection with the invariant subspace problem. Jung, Ko and Pearcy proved that T has a nontrivial invariant subspace if an only if $\Delta\,(T)$ does. On the other hand, Dykema and Schultz proved that the Brown measure is preserved by the Aluthge transform. Another reason is related with the iterated Aluthge transform. Let $\Delta^0\,(T) = T$ and $\Delta^n\,(T) = \Delta\,(\Delta^{n-1}\,(T))$ for every $n \in \mathbb{N}$ Jung, Ko and Peacv raised the following conjecture:

Conjecture 1. The sequence of iterates $\{\Delta^n\,(T)\}_{n\in\mathbb{N}}$ converges, for every matrix T. A

This study intends to give a brief survey on different properties of the Aluthge transform, making special emphasis on those results related with Conjecture 1, which was originally stated for operators on Hilbert spaces, and remains open for finite factors.

We begin the article with a historical summary that helps to explain the connection of the Aluthge transform with the invariant subspace problem and to describe some results that motivated and suggested that the conjecture might be true for operators on Hilbert spaces. Nevertheless, some couterexamples were found in this setting. We will expose one of them with some detail, which is particularly interesting because it shows an operator $T \in L(\mathcal{H})$ such that the sequence $\{\Delta^n\,(T)\}_{n\in\mathbb{N}}$ does not converge even in the weak operator topology.

In 1987, Brown was able to prove that every hyponormal operator whose spectrum has non-empty interior has a non-trivial invariant subspace. In 1990, Aluthge considered the possibility of extending this result to p-hyponormal operator and defined what is now called Aluthge transform. The first result that caught the attention on this transformation is summarized in the following statement:

Theorem 3.2 (Aluthge). Let $T \in L(\mathcal{H})$ be p-hvponormal. Then

- If $p \geq \frac{1}{2}$, then $\Delta(T)$ is hn,
- If $p < \frac{1}{2}$, then $\Delta(T)$ is $(p + \frac{1}{2})$-hn,
- It holds that $\Delta(\,\Delta(T)\,)$ is hn. □

Later on, Jung, Ko and Pearcy proved the next result that allowed to extend Brown's result to p-hyponormal operators:

Theorem 3.3 (Jung-Ko-Pearcv). If Lat(T) denotes the lattice of invariant subspaces of a given operator $T \in L(\mathcal{H})$, then $Lat(T) \simeq Lat(\Delta\,(T))$.
□

This result led to the first version of Jung-Ko-Pearcv conjecture on the iterated Aluthge transform sequence: The sequence of iterates $\{\Delta^n\,(T)\}_{n\in\mathbb{N}}$ converges to a normal operator for every $T \in L(\mathcal{H})$. As soon as they raised this conjecture, many results supporting this conjecture appeared. The following formula for the spectral radius due to Yamazaki was one of the most important:

Theorem 3.4 (Yamazaki). Given $T \in L(\mathcal{H})$, then $\rho(T) = \lim_{n\to\infty} \|\Delta^n(T)\|$.

However, after several positive partial results, some counterexamples appeared. One of the most interesting was found by Yanahida's. Using a smart selection of weights, Yanahida defines a weighted shift operator whose sequence of iterated Aluthge transforms does not converge, even with respect to the weak operator topology! Let us briefly describe it: let $\{e_k\}_{k\in\mathbb{N}}$ be the canonical basis of $\ell^2(\mathbb{N})$, and $T \in L(\ell^2(\mathbb{N}))$ the weighted shift operator defined by $Te_k = a_k e_{k+1}$ where

$$a_{0,k} = a_k = \begin{cases} 1 & \text{if } k \in [4^{2n-1}+1, 4^{2n}] \\ e & \text{if } k \in [0,4] \text{ or } k \in [4^{2n}+1, 4^{2n+1}] \end{cases}$$

Straightforward computations show that $\Delta^m(T)$ is also a weighted shift with weights:

$$a_{m,k} = \prod_{j=k}^{k+m} a_j^{\binom{m}{j}1/2^m}, \quad k \in \mathbb{N}.$$

Then, using some tricky estimates, it can be proved that the sequence $\{\langle \Delta^m(T)e_1, e_2\rangle\}_{m\in\mathbb{N}}$ does not converge, which implies that the sequence of iterates does not converge in the weak operator topology.

CONVERGENCE OF ITERATED ALUTHGE TRANSFORM

Let $\mathcal{H}$ be a complex Hilbert space, and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Given $T \in L(\mathcal{H})$, consider its (left) polar decomposition $T = U|T|$. In order to study the relationship among p-hyponormal operators, Aluthge introduced the transformation $\Delta_{1/2}(\cdot) : L(\mathcal{H}) \to L(\mathcal{H})$ defined by $\Delta_{1/2}(T) = |T|^{1/2}U|T|^{1/2}$. Later on, this transformation, now called Aluthge transform, was also studied in other contexts by several authors, such as Jung, Ko and Pearcy, Ando, Ando and Yamazaki (3|, Yamazaki, Okubo pEj and Wu pRJf among others.

In this study, given $\lambda \in (0,1)$ and $T \in L(\mathcal{H})$, we study the so-called λ-Aluthge transform of T defined by

$$\Delta_\lambda(T) = |T|^\lambda U |T|^{1-\lambda}$$

This notion has already been considered by Okubo. We denote by $\Delta_\lambda^n(T)$ the n-times iterated λ-Aluthge transform of T, i.e.

$$\Delta_\lambda^0(T) = T;$$

and

$$\Delta_\lambda^n(T) = \Delta_\lambda(\Delta_\lambda^{n-1}(T))\, n \in \mathbb{N}. \quad (1.1)$$

In a previous study, we show that the iterates of usual Aluthge transform $\Delta_{1/2}^n(T)$ converge to a normal matrix $\Delta_{1/2}^\infty(T)$ for every diagonalizable matrix $T \in \mathcal{M}_r(\mathbb{C})$ (of any size). We also proved the smoothness of the map $T \mapsto \Delta_{1/2}^\infty(T)$ when it is restricted to a similarity orbit, or to the (open and dense) set $\mathcal{D}_r^*(\mathbb{C})$ of invertible rxr matrices with r different eigenvalues. The key idea was to use a dynamical systems approach to the Aluthge transform, thought as acting on the similarity orbit of a diagonal invertible matrix. Recently, Huajun Huang and Tin-Yau Tam showed, with other approach, that the iterates of every λ-Aluthge transform $\Delta_\lambda^n(T)$ converge, for every matrix $T \in \mathcal{M}_r(\mathbb{C})$ with all its eigenvalues of different, moduli.

CONVERGENCE OF THE NORMS OF DUGGAL ITERATES

We shall prove that $\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = r(T)$ for operators T belonging to certain classes of operators in C(H). By the inequality, $\| \widehat{T}^{(n+1)} \| \le \| \widehat{T}^{(n)} \|$ for all $n \in \mathbb{N}$. Moreover $\sigma(\widehat{T}^{(n)}) = \sigma(T)$, and hence $r(\widehat{T}^{(n)}) = r(T)$ for all $n \ge 0$

Thus $\{ \|\widehat{T}^{(n)}\| \}_{n=0}^\infty$ is a decreasing sequence which is bounded below by $r(T)$ The following lemma is an easy consequence.

Lemma 3.2. There is an $s \ge r(T)$ for which $\lim_{n\to\infty} \| \widehat{T}^{(n)} \| = s$

Remark 3.1. We notice one more analogy between Aluthge and Duggal transformations.

The sequence $\{\| (\widetilde{T}^{(n)}) \|\}_{n=0}^\infty$ is decreasing such that $r(T) \le \| \widetilde{T}^{(n)} \| \le \|T\|$, and $r(\widetilde{T}^{(n)}) = r(T)$ for all n. The sequence $\{\| \ (\widehat{T}^{(n)}) \ \|\}_{n=0}^{\infty}$ is decreasing such that $r(T) \leq \| \ \widehat{T}^{(n)} \ \| \leq \|T\|$, and $r(\widehat{T}^{(n)}) = r(T)$ for all n.

Theorem 3.6 (Mc Intosh inequality). For bounded linear operators A,B and x,

$$\| \ A^*XB \ \| \leq \| \ AA^*X \ \|^{1/2} \ \| \ XBB^* \ \|^{1/2}$$

Theorem 3.7 (Heinz inequality). For positive linear operators A and B, and bounded linear operator X.

$$\| \ A^{\alpha}XB^{\alpha} \ \| \leq \| \ AXB \ \|^{\alpha} \ \| \ X \ \|^{1-\alpha}$$

for all $0 \leq \alpha \leq 1$

Using these inequalities we prove the following results.
Lemma 3.3. For any positive integer k,

$$\| \ (\widehat{T}^{(n+1)})^k \ \| \leq \| \ (\widehat{T}^{(n)})^k \ \|$$

for all $n \geq 0$. Consequently, the decreasing sequence $\{\| \ (\widehat{T}^{(n)})^k \ \|\}_{n=0}^{\infty}$ is convergent.

Proof. Let $f(t) = t^k, t \in$ a neighborhood of $\sigma(T)$, and note that $\sigma(T) = \sigma(\widehat{T}^{(n)})$

We have $f \in Hol(\sigma(T))$.

Lemma 3.4. If $\widehat{T}^{(n)} = U_n|\widehat{T}^{(n)}|$ is the polar decomposition of $\widehat{T}^{(n)}$, then for any positive integer k,

$$\| \ (\widehat{T}^{(n+1)})^k \ \| \leq \| \ |\widehat{T}^{(n)}|^2 \ (\widehat{T}^{(n)})^{k-1} \ \|^{1/2} \ \| \ (\widehat{T}^{(n)})^{k-1} U_n U_n^* \ \|^{1/2}$$

Proof. We have $\widehat{T}^{(n+1)} = |\widehat{T}^{(n)}| U_n$ and therefore $(\widehat{T}^{(n+1)})^k = |\widehat{T}^{(n)}| (\widehat{T}^{(n)})^{k-1} U_n$

Hence by theorem 3.6,

$$\begin{aligned} \| \ (\widehat{T}^{(n+1)})^k \ \| \ &\leq \ \| \ |\widehat{T}^{(n)}| (\widehat{T}^{(n)})^{k-1} U_n \ \| \\ &\leq \ \| \ |\widehat{T}^{(n)}|^2 (\widehat{T}^{(n)})^{k-1} \ \|^{1/2} \ \| \ (\widehat{T}^{(n)})^{k-1} U_n U_n^* \ \|^{1/2} \end{aligned}$$

CONCLUSION

Some of the problems that were thought about and where further research is possible. It is proved in this paper that if *T* is a bounded linear operator on a Hilbert, space, then the sequence of the norms of Duggal iterates of *T* converges to the spectral radius r(T'), if *T* satisfies certain conditions. In general, unlike Aluthge transformations, the sequence need not converge to r(T). One can attempt to characterize operators *T* on a Hilbert space $\mathcal{H}$, having the property that the sequence of the norms of Duggal iterates of *T* converges to the spectral radius *r(T).*

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