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REVIEW ARTICLE

CONSTRUCTIVIST ALTERNATIVES TO SCHOOL MATH

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Constructivist Alternatives to School Math

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INTRODUCTION

Some educators such as Magdalene Lampert (1990) and Deborah Ball (1991b), while at the same time trying to increase classroom safety, have exploited the riskiness of open conjecture in motivating students to defend their ideas in classroom arguments.

Students who have a stake in maintaining their reputation as mathematical conjecturers are creative in finding clever arguments, examples, and counterexamples to support their own ideas and refute others'. In this way, the notion of mathematical proof is constructed as a limiting form of everyday classroom argument. A proof is a very convincing argument, and what counts as convincing depends very much on the sophistication of the community one is attempting to convince.

Another illuminating example of large group mathematical activity is to be found in the doctoral work of Idit Harel (1988). In Harel's project, a fourth grade class embarked on a project to author software for instructing third graders about fractions.

Students were free to work alone or in groups, but in practice new ideas of how to present material circulated in the classroom in a "collaboration in the air" (Harel & Kafai, 1991).

Even children who were most comfortable working alone ended up participating in the creation of a "fractions culture" and children who had been outside the social circle of the classroom enjoyed increased self-esteem as a result of the recognition of their contributions to that culture.

In the past twenty years there has been some (though still not much) movement in the elementary and even high schools toward teaching children to be mathematicians instead of teaching them mathematics (see Papert, 1971). A considerable literature has grown up describing experiments in constructivist teaching in the schools (e.g. Hoyles & Noss, 1992; Steffe & Wood, 1990; Streefland, 1991; Von Glasserfeld, 1991). A plethora of creative approaches have begun to proliferate:

Hoyles and Noss (1992) integrate the computer and the language LOGO into the mathematics classroom, providing new means of expressing mathematics, and encouraging a diversity of styles and approaches to learning/doing mathematics.

Increasing availability of computers in classrooms has facilitated the development of computer microworlds for mathematical explorations (see e.g. Edwards, 1992;

Schwartz & Yerushalmy, 1987; Leron & Zazkis, 1992) and emphasized dynamic mathematical processes as opposed to static structures. (see e.g. Vitale, 1992).

Strohecker (1991) explores ideas of mathematical topology through classroom discussions of and experiments with tying knots.

Cuoco & Goldenberg (1992) focus on geometry as providing a "natural" bridge between the visual intuitions of the high schooler and more symbolic areas of mathematics and, by so doing, "expands students' conception of mathematics itself."

Confrey (1991) points out how easily we fall into the misconceptions trap and fail to listen to the sense, the novel meaning making of the mathematical learner.

Schoenfeld (1991) moves from a college problem solving class using pre-decided problems to a focus on learners' creation of related problems and, ultimately, to the development of classrooms that are "microcosms of mathematical sense making".

Konold (1991) works with beginning college students on probability and focuses on how students interpret probability questions. He concludes that effective probability instruction needs to encourage learners to pay attention to the fit between their intuitive beliefs and 1) the beliefs of others 2) their other related beliefs and 3) their empirical observations.

Rubin & Goodman (1991) have developed video as a tool for analyzing statistical data and supporting statistical investigations. Nemirovsky (1993) & Kaput (in press) have developed curricular materials for exploring a qualitative calculus. They view themselves as part of a calculus reform movement that seeks to understand calculus as the mathematics of change and thus make calculus into a "fundamental longitudinal strand" for the learning of mathematics.

Most of the above alternatives took place in the elementary school setting, and even those that took place in high school or college tended to focus on relatively elementary mathematics and relatively novice learners. As of yet, there has been very little written about the methods used at higher educational levels (late college through graduate school through life beyond school) which if anything are more rigidly instructionist (see Papert, 1990), provide less opportunity for self-directed exploration and conceptual understanding than do classes in the lower grades.

CONCRETE LEARNING

"No ideas but in things" - William Carlos Williams

I will now advance a theoretical position which is the underlying force behind this study. This will involve a critical investigation of the concept of "concrete" and what it means to make mathematics concrete.

Concepts that are said to be concrete are also said to be intuitive - we can reason about them informally. It follows that if we hold the standard view that concreteness is a property of objects which some objects have and some do not, we are likely to be led into the view that our intuitions are relatively fixed and unchangeable. This last conclusion, that of the relative unchangeability of intuitions, is the force behind the prescriptions to avoid intuitive probability judgments. When we replace the standard view of concrete with a new relational view, we also see the error in the static view of intuitions. In understanding the process of developing concrete relationships we learn how to support the development of mathematical intuitions.

In the present chapter, I will expose the weakness of the standard view of concrete, and probe the very notion of "concreteness". I propose a new definition of "concrete". In the new model, concreteness is a property not an object itself but of our relationship to that object. A concept is "concrete" when

- it is well-connected to many other concepts
- we have multiple representations of it
- we know how to interact with it in many modalities.

As such, the focus is placed on the process of developing a concrete relationship with new objects, a process I call "concretion".

Implications for pedagogy of this new view of concrete will be explored, and a multi-faceted analogy will be constructed with conservation experiments of Piaget.

Many of these ideas are expounded in greater detail in (Wilensky, 1991a).

CONCRETION

Seymour Papert has recently called for a "revaluation of the concrete": a revolution in education and cognitive science that will overthrow logic from "on top and put it on tap." Turkle and Papert (1991) situate the concrete thinking paradigm in a new "epistemological pluralism"-- an acceptance and valuation of multiple thinking styles, as opposed to their stratification into hierarchically valued stages. As evidence of an emerging trend towards the concrete, they cite feminist critics such as Gilligan's (1982) work on the contextual or relational mode of moral reasoning favored by most women, and Fox Keller's (1983) analysis of the Nobel-prize-winning biologist Barbara McClintock's proximal relationship with her maize plants, her "feeling for the organism."

They cite hermeneutic critics such as Lave (1988), whose studies of situated cognition suggest that all learning is highly specific and should be studied in real world contexts.

For generations now we have viewed children's intellectual growth as proceeding from the concrete to the abstract, from Piaget's concrete operations stage to the more advanced stage of formal operations (e.g., Piaget, 1952). What is meant then by this call for revaluation of the concrete?

And what are the implications of this revaluation for education? Are we being asked to restrict children's intellectual horizons, to limit the domain of inquiry in which we encourage the child to engage? Are we to give up on teaching general strategies and limit ourselves to very context specific practices? And what about mathematics education? Even if we were prepared to answer in the affirmative to all the above questions for education in general, surely in mathematics education we would want to make an exception? If there is any area of human knowledge that is abstract and formal, surely mathematics is. Are we to banish objects in the head from the study of mathematics? Should we confine ourselves to manipulatives such as Lego blocks and Cuisinaire Rods? Still more provocatively, shall we all go back to counting on our fingers?

The phrases "concrete thinking", "concrete-example", "make it concrete" are often used when thinking about our own thinking as well as in our educational practice.

To begin our investigation we will need to take a philosophical detour and examine the meaning of the word concrete. What do we mean when we say that something--a concept, idea, piece of knowledge (henceforward an object)--is concrete?

STANDARD DEFINITIONS OF CONCRETE

Our first associations with the word concrete often suggest something tangible, solid; you can touch it, smell it, kick it ; it is real. A closer look reveals some confusion in this intuitive notion. Among those objects we refer to as concrete there are words, ideas, feelings, stories, descriptions. None of those can actually be "kicked." So what are these putative tangible objects we are referring to?

One reply to the above objection is to say: No no, you misunderstand us, what we mean is that the object referred to by a concrete description has these tangible properties, not the description itself. The more the description allows us to visualize (or, if you will, sensorize) an object, to pick out, say, a particular scene or situation, the more concrete it is. The more specific the more concrete, the more general the less concrete. In line with this, Random House says concrete is "particular, relating to an instance of an object" not its class.

Let us call the notion of concrete specified by the above the standard view.

According to the standard view, concrete objects are specific instances of abstractions, which are generalities. Concrete objects are rich in detail, they are instances of larger classes; abstract objects are scant in detail, but very general.

Given this view, it is natural for us to want our children to move away from the confining world of the concrete, where they can only learn things about relatively few objects, to the more expansive world of the abstract, where what they learn will apply widely and generally. Yet somehow our attempts at teaching abstractly leave our expectations unfulfilled. The more abstract our teaching in the school, the more alienated and bored are our students, and far from being able to apply their knowledge generally across domains, their knowledge displays a "brittle" character, usable only in the exact contexts in which it was learned.

Numerous studies have shown that students are unable to solve standard math and physics problems when these problems are given without the textbook chapter context. Yet they are easily able to solve them when they are assigned as homework for a particular chapter of the textbook (e.g., DiSessa, 1983; Schoenfeld, 1985).

CRITIQUES OF THE STANDARD VIEW

Upon closer examination there are serious problems with the standard view. For one thing, the concept of concrete is culturally relative. For us, the word "snow" connotes a specific, concrete idea. Yet for an Eskimo, it is a vast generalization, combining together twenty-two different substances. As was noted by Quine (1960), there are a multitude of ways to slice up our world, depending on what kind and how many distinctions you make. There is no one universal ontology in an objective world.

An even more radical critique of the notion of a specific object or individual entity comes out of recent research in artificial intelligence (AI). In a branch of AI called emergent AI, objects that are typically perceived as wholes are explained as emergent effects of large numbers of interacting smaller elements. Research in brain physiology as well as in machine vision indicate that the translation of light patterns on the retina into a "parsing" of the world into objects in a scene is an extremely complex task. It is also underdetermined; by no means is there just one unique parsing of the inputs. Objects that seem like single entities to us could just as easily be multiple and perceived as complex scenes, while apparently complex scenes could be grouped into single entities. In effect the brain constructs a theory of the world from the hints it receives from the information in the retina. It is because children share a common set of sensing apparatus (or a common way of obtaining feedback from the world, see Brandes & Wilensky, 1991) and a common set of experiences such as touching, grasping, banging, ingesting , that children come as close as they do to "concretizing" the same objects in the world.

The critique of the notion of object connects also to the work of Piaget. In Piaget's view of development, the child actively constructs his/her world. Each object constructed is added to the personal ontology of the child. Thus we can no longer maintain a simple sensory criterion for concreteness, since virtually all objects, all concepts which we understand, are constructed, by an individual, assembled in that particular individual's way, from more primitive elements. Objects are not simply given to the senses; they are actively constructed.

We have seen that when we talk about objects we can't leave out the person who constructs the object. It thus follows that it is futile to search for concreteness in the object -- we must look at a person's construction of the object, at the relationship between the person and the object.

TOWARDS A NEW DEFINITION OF CONCRETE

I now offer a new perspective from which to expand our understanding of the concrete. The more connections we make between an object and other objects, the more concrete it becomes for us. The richer the set of representations of the object, the more ways we have of interacting with it, the more concrete it is for us. Concreteness, then, is that property which measures the degree of our relatedness to the object, (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, the quality of our relationship with the object.

Once we see this, it is not difficult to go further and see that any object/concept can be become concrete for someone. The pivotal point on which the determination of concreteness turns is not some intensive examination of the object, but rather an examination of the modes of interaction and the models which the person uses to understand the object. This view will lead us to allow objects not mediated by the senses, objects which are usually considered abstract—such as mathematical objects—to be concrete; provided that we have multiple modes of engagement with them and a sufficiently rich collection of models to represent them.

When our relationship with an object is poor, our representations of it limited in number, and our modes of interacting with it few, the object becomes inaccessible to us.

So, metaphorically, the abstract object is high above, as opposed to the concrete objects, Seymour Papert has said: “You can’t think about thinking without thinking which are down and hence reachable, “graspable.” We can dimly see it, touch it only with removed instruments, we have remote access, as opposed to the object in our hands that we can operate on in so many different modalities. Objects of thought which are given solely by definition, and operations given only by simple rules, are abstract in this sense.

Like the word learned only by dictionary definition, it is accessible through the narrowest of channels and tenuously apprehended. It is only through use and acquaintance in multiple contexts, through coming into relationship with other words/concepts/experiences, that the word has meaning for the learner and in our sense becomes concrete for him or her. As Minsky says in his Society of Mind:

The secret of what anything means to us depends on how we've connected it to all the other things we know. That's why it's almost always wrong to seek the “real meaning” of anything. A thing with just one meaning has scarcely any meaning at all (Minsky, 1987 p. 64).

This new definition of concrete as a relational property turns the old definition on its head. Now, thinking concretely is seen not to be a narrowing of the domain of intellectual discourse, but rather as opening it up to the whole world of relationship. What we strive for is a new kind of knowledge, not brittle and susceptible to breakage like the old, but in the words of Mary Belenky, “connected knowing“ (Belenky, Clinchy, Goldberger, & Tarule, 1986).

CONSEQUENCES OF THE NEW VIEW

Below is a pithy summary of what I see as the consequences, both theoretical and practical, of the above reformulation of the concrete/abstract dichotomy:

• It's a mistake to ask what is the "real meaning" of a concept.

This "thing with only one meaning" is a characterization of a formal mathematical definition. An object of thought which is given solely by definition, and an operation given only by simple rules, is abstract in this sense. Like the word learned only by dictionary definition, it is accessible through the narrowest of channels and tenuously apprehended. It is only through use and acquaintance in multiple contexts, through coming into relationship with other words/concepts/experiences, that the word has meaning for the learner and in our sense becomes concrete for him or her.

• When concepts are very robust they become concrete.

From Piaget's conservation experiments, we see that what appears to us to be a very concrete property of an object, like its very identity, is constructed by the child over a long developmental period. After a brief period of transition, when the child acquires the conserved quantity, she sees it as a concrete property of the object.

Similarly, a machine vision program must build up notions of objects from more primitive notions in its vision ontology. Which things it adds to its ontology, making them concrete "wholes", depends on what primitive elements it started with and what objects it has already built up. As Quine has shown in his famous "rabbits vs. undetached rabbit-parts" example (Quine, 1960), there is no way unique way to slice up the world into objects.

• There's no such thing as abstract concepts - there are only concepts that haven't yet been concretized by the learner.

We mistakenly think that we should strive for abstract knowledge since this seems to be the most advanced kind of knowledge that we have. The fallacy here is that those objects which we have abstract knowledge of are those objects that we haven't yet concretized due to their newness or difficulty to apprehend. When we strive to gain knowledge of these, it seems like we strive for the abstract. But in fact, when we have finally come to know and understand these objects, they will become concrete for us.

Thus Piaget is stood on his head: development proceeds from the abstract to the concrete.

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