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Configuration of Groups Containing Generalized Normal Subgroups

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Abstract: This paper aims to identify unique symmetric presentation of normal subgroups. It is often observed that a subgroup X of a group G is considered almost normal if the index |G:NG(X)| is finite, while X is termed nearly normal if it possesses a finite index in its normal closure . We introduce some concepts related to commutators and abelian subgroups. This paper explores the structure of groups in which every infinite subgroup is either almost normal or nearly normal.

Keywords: Commutator, Abelian subgroups, Formalizers, Normal Subgroup, Finite subgroups

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INTRODUCTION

This chapter provides a short mathematical introduction to group and subgroup algebra. It is constructed in the following order: definitions, propositions, and proofs. The concepts and terminology provided here will serve as the foundation for the chapters that follow, which will deal with group theory in its tighter sense and its application to physics problems. Mathematics prerequisites are at the bachelor's level. 1 A subgroup X of a group G is said to be virtually normal if it contains a finite number of conjugates, or equivalently, if the normaliser NG(X) of X has a finite index in G. B.H. Neumann famously proved that all subgroups of a group G are practically normal if and only if the centre Z(G) has a finite index in G. I.I Eremin [4] expanded on this theory, demonstrating that a subgroup X of a group G is nearly normal if it has a finite index in its normal closure, . In the aforementioned publication, Neumann demonstrated that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. It is sufficient to assume that all abelian subgroups are nearly normal (a result that was obtained by M.J.Tomkinson [5]). In general, the concepts of almost normal and nearly normal subgroups are incomparable. For instance, all subgroups of the base group of the standard wreath product W= are almost normal in W, but only a few of them are nearly normal. Similarly, if G is any group containing an infinite minimal normal subgroup N, which is abelian of prime exponent, then each proper subgroup of finite index of N is nearly normal in G but has infinitely many conjugates.On the other hand, it follows from Neumann's work and the renowned theorem of Schur on the finiteness of the commutator subgroup of central-by-finite groups (see, for example,[7] Part 1, Theorem 4.12) that if all subgroups of a group G are almost normal, then they must be nearly normal.It should also be noted that the structures of groups in which every infinite subgroup is almost normal and groups in which all infinite subgroups are nearly normal have been discussed separately. It turns out that knowing that all subgroups of a certain type are virtually normal (or that they are all almost normal) is useful.The major purpose of this paper is to analyse groups in which every member of a relevant system of subgroups is either almost normal or nearly normal; in particular, Tomkinson's theorem mentioned above will be generalised to this case.

Y.D. Polovicki result demonstrates that a group G has finitely many normalisers of abelian subgroups if and only if it is finite in the centre. In recent years, many more studies have appeared on the structure of groups with finitely many normalisers of subgroups with a certain property (see for example [2],[8],[9],[10],[19]); We will also look at groups with a finite number of normalisers for subgroups that are neither virtually normal nor nearly normal.

PRELIMINARIES

Definition : A normal subgroup of a group G is defined as $gng^{-1} \in N$ for each $g \in G$ and

$n \in N$. We designate it as $N < G$. It can be observed that every subgroup N of an abelian group G is a normal subgroup of G, for $g \in G$ and $n \in N$.

$\Rightarrow gn = ng \Rightarrow gng^{-1} = n \in N$

From the above observation ,it follows that every subgroup of a cyclic group is normal ,since a cyclic group is abelian.

Example: Let $G = \langle a \rangle$, $a^{12} = e$ then $H_1 = \langle a^2 \rangle = \{a^2, a^4, a^6, a^8, a^{10}, a^{12} = e\}$

$H_2 = \langle a^3 \rangle = \{a^3, a^6, a^9, , a^{12} = e\}$

$H_3 = \langle a^4 \rangle = \{a^4, a^8, a^{12} = e\}$

$H_4 = \langle a^6 \rangle = \{a^6, a^{12} = e\}$.

Example: $N = \{1, -1\}$ is normal subgroup of the multiplicative group.

$G = \{1, -1, i, -i\}$.

Example: If N and M are normal subgroups of a group G, then $N \cap M$ is normal subgroup of G.

Since N and M are subgroups of G ,so $N \cap M$ is a subgroup of G .Let $g \in G$ and $a \in N \cap M$.so that $a \in N$ and $a \in M$.

Since $N < G$, $gag^{-1} \in N$

Since $M < G$, $gag^{-1} \in M$.

$\therefore gag^{-1} \in N \cap M \; \forall \; g \in G$ and $a \in N \cap M$.

Example: If H is a subgroup of G and N is a normal subgroup of G, then H ∩ N is a normal subgroup of H.

Since H ∩ N ⊂ H, H ∩ N is a subgroup of H.

Now we show that H ∩ N < H

Let h ∈ H and a ∈ H ∩ N (i.e a ∈ G and a ∈ N)

We shall show that hah^{-1} ∈ H ∩ N .

Since H < G , $h^{-1} \in H$ and a ∈ H ⇒ha$h^{-1} \in H$

Since N < G ,so h ∈ H and a ∈ N ⇒ hah^{-1} ∈ N.

Now hah^{-1} ∈ H and hah^{-1} ∈ N

⇒ hah^{-1} ∈ H ∩ N.

Hence H ∩ N is a normal subgroup of H.

Example: If N and M are normal subgroups of a group G and if N ∩ M = (e) ,then nm = mn for each n ∈ N and m ∈ M.

Theorem: Prove that H is a normal subgroup of a group G iff the product of any two right cosets of H in G is a right coset of H in G .

Proof: The condition is necessary .

Let H< G and Ha ,Hb be any two right cosets of H in G (a,b ∈ G)

We shall prove that haHb is a right coset.

We have Ha Hb = H(aH)b = H (Ha) as H< G.

= HH ab = Hab as HH =H (' ∵ $H <$ G)

Thus HaHb = Hab ,(ab ∈ G) shows that the product of two right cosets Ha and Hb is the right coset Hab.

The condition is sufficient .

Suppose the product of two right cosets of H in G is a right coset of H in G.

Let HaHb =Hc ; a,b ,c ∈ G

We shall prove that H < G.

Since a =ea ∈ Ha and b = eb ∈ Hb , so ab ∈ Ha Hb = Hc

Now ab ∈ Hc ⇒ Hab ∀ a,b ∈ G

Let x ∈ G and h ∈ H.

Then x ∈ Hx , $h \in H$ = He and $x^{-1} \in H\,x^{-1}$

⇒xh$x^{-1} \in HxHeHx^{-1}$

⇒xh$x^{-1} \in HxeHx^{-1}$ = HxHx^{-1} =He =H

⇒ xh$x^{-1} \in H\, \forall\, x \in G\,, h \in H$. *Hence* $H < G$.

ABELIAN SUBGROUP

Remember that the FC-center of a group G is the subgroup that contains all elements with finitely many conjugates, and G is a F C -group if it coincides with its FC-Centre. The theory of FC-groups is relevant in many concerns about infinite groups with finiteness criteria; the monograph contains the main features of FC-groups.

It is easy to show that a cyclic subgroup of a group G is almost normal if and only if it is nearly normal, and both such features are also equivalent to the fact that the conjugacy class of x in G is finite. Thus a group G is an FC-group if and only if all its cyclic subgroups are almost normal(or nearly normal). It turns out that in the universe of FC-groups, approximately normalcy is a stronger feature than nearly normality.

Lemma: Let G be an FC-group, and let X be a nearly normal subgroup of G. Then X is approximately normal in G.

Proof: As the normalize NG(X) has finite index in G ,there exits a finitely generated subgroup E of G such that G =hE, NG(X) i. Moreover ,the subgroup E can be chosen normal in G, since its normal closure EG is like –wise finitely generated .Thus G = NG(X) E and so XG =XE. Clearly ,E is central –by-finite ,so that it satisfies the maximal condition on subgroups and in particular its subgroup[X,E] is finitely generated .On the other hand , the commutator subgroup of G is locally finite and hence [X,E] is finite .Therefore the subgroup X has finite index in its normal closure XG =X[X,E] and so it is nearly normal in G.

The main result of this section is an extension of Tomkinson's theorem from the introduction to the case in which every (abelian) subgroup is either almost normal or nearly normal; it will be obtained as a result of a theorem on groups with finitely many normalisers of subgroups with a suitable property. For this reason, we require a result of B.H. Nesumann, which holds in the more general case of groups covered by cosets of subgroups.

The main result of this section is an extension of Tomkinson's theorem quoted in the introduction to the case in which every (abelian) subgroup is either almost normal or nearly normal; it is obtained as a result of a theorem on groups with finitely many normalisers of subgroups with a suitable property. For this reason, we need B.H.Nesumann's conclusion, which holds in the more general case of groups covered by subgroup cosets.

Lemma : Consider the group G = $X_1 \cup \ldots \cup X_t$ as the union of finitely many subgroups $X_1, \ldots, X_t$ .Then all $X_i's$ is with infinite indexes can be excluded from this decomposition, specifically at least one of. The subgroups $X_1, \ldots, X_t$ have a finite index in G.

Theorem: Let G be a group with a finite number of normalisers for abelian subgroups that are neither virtually normal nor nearly normal. The commutator subgroup G' of G is finite, implying that all G subgroups are roughly normal.

Proof: Let, $\{N_G(X_1), \ldots\ldots N_G(X_k)\}$

be the the set of normalizers of abelian subgroups of G which are neither almost normal nor roughly normal .If x is any element of G having infinitely many conjugates, we have

$N_G(\langle x\rangle) = N_G(X_j)$ for some j $\leq k$, so that in particular x belongs to $N_G(X_j)$ and hence

$$G = F \cup N_G(X_1) \cup \ldots \cup N_G(X_k),$$

Where F is the FC-centre of G. On the other hand ,each normalize $N_G(X_i)$ has infinite index in G, and so it follows from Lemma 2.2 that G = F is an FC-group .The same result shows that the set

$$N_G(X_1) \cup \ldots \cup N_G(X_k),$$

Is properly contained in G, so that we may consider an element g of

$$G \setminus \left(N_G(X_1) \cup \ldots \cup N_G(X_K)\right).$$

Clearly, g normalizes all subgroups of its centralizer $C_G(g)$, so that each abelian subgroup of $C_G(g)$ must be either almost normal or nearly normal. Therefore all abelian subgroups of $C_G(g)$ are nearly normal by lemma 2.1 ,and hence $C_G(g)$ has finite commutator subgroup by Tomkinson's theorem. As the index $|G : C_G(g)|$ is finite, the group G is finite –by –abelian –by –finite. On the other hand, any abelian-by –finite FC-group is central-by-finite, and so it follows from Schur's theorem that the commutator subgroup G' of G is finite. In particular, all subgroups of G are nearly normal.

COMMUTATOR SUBGROUPS

Definition: (Subgroup Generated by a set): Let S be a nonempty subset of the group G. The smallest subgroup of G containing S is known as the S-generated subgroup. It is represented by ⟨S⟩. In other words, a non-empty subset S of G generates a subgroup H of G, if

(i) S $\subseteq H$

(ii) If K is any subgroup of G such that S ⊆ K , then H ⊆ K .We write H = $\langle S\rangle$.

Theorem: If S is any non-empty subset of a group G and If H = $\{a \in G : a = s_1, s_2 \ldots\ldots s_n\}$; $s_i \in S$ for each i } ,n being a positive integer ,then H = $\langle S\rangle^i$

Proof: Since S is non-empty ,so H is non-empty . Also S ⊆ H.

Let $\alpha, \beta \in H$. Then

$\alpha = S_1 S_2 \dots S_n$; S_i or $s_i^{-1} \in$ S for each i.

$\beta = t_1 t_2 \dots t_m$; t_j or $t^{-1}{}_j \in S$ for each j.

$\therefore \alpha\beta^{-1} = s_1 s_2 \dots s_n t_m^{-1} \dots t_2^{-1} t_m^{-1} \in$ H.

Thus H is a subgroup of G containing S. Let K be any subgroup of G such that S ⊆ K.

Then for each s $\in S$, s ∈ K and so $s^{-1} \in$ K (∵ K $< G$) ….(i)

Now we show that H ⊆ K .

Let $\alpha \in$ H so that $\alpha = s_1 s_2 \dots s_n$ where s_i or $s_i^{-1} \in$ S for each i. …..(ii)

From (i) and (ii) we get $\alpha \varepsilon K$ (∵ K $< G$) ∴ H ⊆ K.

Hence H is the subgroup of G generated by S

i.e H = $\langle S \rangle$

Definition: (Commutator) : If x and y are any two elements of a group G, then $xyx^{-1}y^{-1}$ is called the commutator of x and y .

Remark: It may be observed that inverse of a commutator is a commutator ,since

$$\alpha = xyx^{-1}y^{-1} \Rightarrow \alpha^{-1} = (xyx^{-1}y^{-1})^{-1}$$

$$= (y^{-1})^{-1}(x^{-1})^{-1}y^{-1}x^{-1}.$$

$$\therefore \alpha^{-1} = yxy^{-1}x^{-1} \text{ which is commutator}$$

Definition: (Commutator Subgroup): Let S = $\{xyx^{-1}y^{-1} : x, y \in G\}$ be the set of all commutators of a group G. Then the subgroup of G generated by S is called the commutator subgroup of G .It is denoted by G'.Thus $G' = \langle S \rangle$

Example: If H and K are subgroups of a group G, then show that H ⊆K $\Rightarrow H' \subseteq K'$.

Let $\alpha \in H'$ .Then $\alpha = c_1 c_2 \dots c_n$,where each c_i is a commutator of the form

$$c_i = a_i b_i a_i^{-1} b_i^{-1} \; ; \; a_i, b_i \in H$$

Since H is a subgroup of G , c_i ∈ H and so $c_i \in K$ ($\because H \subseteq K$)

Thus each c_i is a commutator in K and so $\alpha = c_1 c_2 \dots c_n \in K'$

Example: If N is a normal subgroup of a group G and N∩ G'= $\{e\}$

Show that N ⊆ Z(G) ,the centre of G.

Let n∈ N and x ∈ G be arbitrary.

Then $xnx^{-1} n^{-1} \in G'$.

Now N < G ⇒ $xnx^{-1} \in G \Rightarrow xnx^{-1}n^{-1} \in$ N.

∴ $xnx^{-1}n^{-1}$=e $\forall x \in G$

⇒ xn = nx $\forall x \in G$

⇒ n ∈ Z(G) by definition of centre of G .

Hence N ⊆ Z(G) .

INFINITE SUBGROUPS OF LOCALLY FINITE GROUPS

Section 4 stated that any virtually normal subgroup of an FC-Group is nearly normal. Our next lemma demonstrates that the contrary result holds in the case of groups with finite Prufer rank.

Lemma: Let G be a group and A be an infinite abelian normal subgroup with a finite index of G. If A is the direct product of a collection of prime order subgroups, then there exists a sequence (an) n $\in N$ of elements of A such that h$ha_n|n \in Ni^G = Dr_{n\in N}$ h$a_n i_G$

Proof: Let a_1 be any non-trivial element of A, and suppose by induction that elements $a_1, \ldots. a_n$ of A have been chosen in such a way that

h$a_n, \ldots, a_n i_G$X… Xh$a_n i_G$

clearly the subgroup h$a_1, \ldots\ a_n i^G$ is finite and hence A contains an infinite subgroup B such that

A = h$a_1 \ldots a_n i^G XB$

Since B has finitely many conjugates in G its core BG has finite index in A and so it is infinite .If a_n+1 is any non-trivial element of BG ,we have

h$a_1, \ldots\ a_n, a_{n+1} i^G\ =\ ha_1 i^G X \ldots Xha_n i^G Xha_{n+1} i^G$

Hence lemma has proved.

Lemma: Let G be a Group and let X be a nearly normal subgroup of G. If X has finite Prufer rank ,then it is nearly normal in G.

Proof: Let G be a group and let X be a nearly normal subgroup of G. Put $|XG: X| = n .Then$ the subgroup (XG)n! is contained in X and hence the factor group XG/ XG has finite exponent. Moreover XG/XG obviously is residually finite and has finite Prufer rank so that a relevant result by Mnn and Segal , Theorem applied to show that XG/XG is locally finite. On the other hand any abelian subgroup of XG/XG is finite and hence it follows from the Hall – Kulatilaka –Kargapolov theorem that XG/XG itself is finite (see for instance Part 1 ,Theorem 3.43) .Therefore X is nearly normal in G .

To analyse groups with finitely many normalisers of finite subgroups that are neither almost normal nor nearly normal, a series of lemma is required.

Lemma: Consider G to be a group with finitely many normalisers of infinite subgroups that are neither virtually normal nor nearly normal. Then G has a characteristic subgroup M of finite index such that the normalizer N_M(X) is normal

in M for each infinite subgroup X of M that is neither almost normal nor near normal in G.

Proof: If X is an infinite subgroup of G that is neither almost normal nor nearly normal, the Normalizer N_G(X) has finitely many images under G automorphisms; in particular, the subgroup N_G(X) has finitely many conjugates in G, therefore the index |G:N_G (N_G(X))| is finite. This also applies to the attributes subgroup.

$$M(X) = \bigcap_{\alpha \epsilon Aut\ G} N_G\left(N_G(X)\right)^{\alpha}$$

has finite index in G .Let A be the set of all infinite subgroups of G which are neither almost normal nor nearly normal. If X and Y are elements of A such that N_G(X) = N_G(Y) ,then M(X) =M(Y) and hence also

$$M = \bigcap_{X \in A} M(X)$$

is a characteristic subgroup of finite index of G .Let X be any infinite subgroup of M which is neither almost normal nor nearly normal. Then

M ≤ M(X) ≤ N_G ($N_G(X)$)

and so the normalizer N_M(X) = N_G(X) ∩ M is a normal subgroup of M.

finally,we mention the following result of D.I. Zaicev [19] ,which is needed in our proofs.

Corollary: Let G be a locally finite group, and let g be an element of G such that X^g ≠ X for any infinite subgroup X of G that is neither virtually normal nor nearly normal. If g has an unlimited number of conjugates, the centraliser C_G (g) is a Cenikov group.

Proof: It follows from the Lemma. Let G be a locally finite group, and let g be an element of G such that X^g= X for any infinite subgroup X of G that is neither virtually normal nor nearly normal. If g has an unlimited number of conjugates, then all abelian hgi-invariant subgroups of G meet the minimum condition that they are also abelian subgroups of the centraliser C_G(g). As a result, C_G (g) meets the minimum condition on abelian subgroups, indicating that it is a Cernikov group according to an important Sunkov result.

To show that locally finite groups in our scenario are near to being locally soluble, we need B.Hartly's result on locally finite groups admitting an automorphism of prime-power order with a small collection of fixed points.

Lemma: Assume G is a locally finite group with finitely many normalisers of infinite subgroups that are neither virtually normal nor nearly normal. The commutator subgroup G' of G must be finite, otherwise G is a Cernikov group with infinite subgroups that are almost normal.

CONCLUSION

Thus, a subgroup X of a group G is almost normal if its index /G: NG(X) is finite, but X is nearly normal if it has a finite index in the normal closure XG. This work looks at the structure of groups in which each (infinite) subgroup is either almost normal or nearly normal.

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