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The addition of Hypergeometric Series and a novel definition of the H function

Akanksha Srivastava [1] * , Dr. Neelam Pandey [2]

1. Research Scholar, Department of Mathematics, Government Model Science College, Rewa, Madhya Pradesh, India

akansha1593@gmaii.com ,

2. Professor, Department of Mathematics, Government Model Science College, Rewa, Madhya Pradesh, India

Abstract: In this study, a new identity for the H function, which is a generalization of a number of special functions, is derived and its use in the summation of the hypergeometric series is discussed. The H function, which belongs to a class of rather wide functions, plays an immense role in complicated math problem solving throughout wide ranges of science and engineering. Thus, obtaining this new identity, we describe a more effective approach to the estimation of hypergeometric series which find numerous applications in combinatorics, physics, and mathematical analysis. The proposed identity reduces existing summation techniques into a single powerful tool for researchers using special functions. Also, this study contrasts the new identity with the conventional methods and demonstrates the superiority of the former in terms of computational efficiency and versatility. The results open the further development of theory of special functions and their usage for solving practical problems.

Keywords: H function, hypergeometric series, summation identity, special functions, mathematical analysis

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INTRODUCTION

The H-function, which is often referred to as the Fox H-function, is a specialized mathematical function that generalizes a number of well-known functions, including the Meijer G-function, hypergeometric functions, and Mittag-Leffler functions, among others (Mori and Morita, 2016). The H-function was first presented by Charles Fox in the early 1960s, and since then, it has developed into a significant instrument in the field of applied mathematics owing to its wide potential for all-encompassing generalization. For example, it is very helpful in the process of addressing difficult issues in a variety of domains, including engineering, physics, and statistical theory (Ono and Rolen, 2013).

Definition and Properties of the H-Function

The H-function is a major expansion within the family of special functions. It encompasses a variety of hypergeometric functions and provides a wide range of applications in the fields of mathematics and physical sciences (Mishra, 2013). Statistical distributions, differential equations, and complex integral solutions are some of the domains in which its use is especially noteworthy. A framework that unifies a variety of functions and enables complicated expressions to be expressed in closed forms is provided by the H-function. This framework makes it possible to conduct theoretical investigation while also improving computing efficiency (Gonzalez-Gaxiola and Santiago, 2012).

Definition of the H-Function-

The H-function, introduced by Fox, is defined in terms of a Mellin-Barnes integral as follows:

$$H_{p,q}^{m,n}(z) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)} z^{-s}\, ds \tag{1}$$

where:

- p, q, m, n are integers with $0 \le m \le q$ and $0 \le n \le p$.
- aj,bj are real or complex constants, and
- Aj,Bj are positive constants.

In most cases, the integration route L is selected due to its ability to effectively separate the poles of Γ(bj−Bjs) and Γ(1−aj+Ajs) in a manner that guarantees convergence (Hahn, 2003). With the use of this formulation, the H-function is able to generalize a broad range of functions, such as the Meijer G-function, the generalized hypergeometric function, and many more, by picking certain parameter values. The fact that it has this quality makes it very adaptable and capable of spanning a wide range of applications across a variety of fields (Denis et al., 2011).

HYPERGEOMETRIC SERIES

A hypergeometric series is a form of power series that generalizes a number of key functions in mathematics. These functions include polynomials, exponential functions, and trigonometric functions, among others. When a hypergeometric series is expressed in its classical form, it is given by (Kexue and Jigen, 2011):

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \tag{2}$$

where (a)n is the Pochhammer symbol, defined as:

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{3}$$

A complex variable is denoted by the letter z, and the parameters a1, a2,..., ap and b1, b2,..., bq are capable of taking on any complex value. In this context, the parameters p and q represent the number of upper and lower parameters, among other things (Lovejoy, 2010).

Types of Hypergeometric Series-

1. **Generalized Hypergeometric Series:** A generalized hypergeometric series is a kind of hypergeometric series that incorporates all hypergeometric series. This type of series is created when

p and q are subject to any non-negative integer values.

2. **Regular Hypergeometric Series:** This category encompasses certain instances in which the parameters are either integers or precise fractions, resulting in functions that have been well researched, such as the Gaussian hypergeometric function.

3. **Confluent Hypergeometric Series:** It is possible to get this form when one of the parameters approaches a limiting case, which ultimately results in the confluent hypergeometric function, which is frequently referred to as

Convergence Conditions-

When it comes to hypergeometric series, the convergence is mostly determined by the values of the parameters and the variable x. There is a complete convergence of the series if:

- z is within the unit circle, i.e., $|z|<1$, for most cases.
- For certain values of parameters, the series may converge at points on the boundary $|z|=1$.

The behavior of convergence might shift depending on the values of p and q used in the calculation. If q is more than p, the series will normally converge for all z, however if q is equal to p, the convergence will often be limited to $|z|<1$ (Laughlin, 2008).

CURRENT IDENTITIES INVOLVING THE H-FUNCTION

An essential aspect in mathematical analysis is played by the H-function, which was first presented as a generalization of a large number of special functions. This function is especially important in the study of differential equations, integral transformations, and probability theory. The H-function is an advanced function that incorporates a wide range of specialised functions, including hypergeometric functions, Meijer G-functions, and others. As a result, its usefulness is broadened to span a variety of domains. Mathematicians have devised various identities incorporating the H-function in order to take use of its adaptability. These identities have made it easier to summaries and convert series, resolved complicated integrals, and offered answers to a wide variety of practical issues (Ahmad, 2008). The summation and transformation formulae represented by the H-function are an important category of identities for the H-function. These identities make it easier to evaluate complicated series and integrals that use the H-function, which would otherwise be difficult owing to the fact that it is dependent on several parameters (Andrews and Warnaar, 2007). The H-function, for example, may be expressed as a finite or infinite series of smaller functions, which makes it easier to estimate or calculate for practical applications. This is an example of a summation identity that is often employed. In addition, expansions of the H-function in terms of other special functions, such as Bessel functions or generalized hypergeometric functions, are included in this category of identities (Bringmann and Ono, 2007). These identities enable mathematicians to use known characteristics and computational methods in order to solve equations that include the H-function. This is accomplished by breaking the H-function down into series that involve these well-known functions (Bringmann and Ono, 2007).

In addition to this, the integral representations of the H-function provide still another crucial identity.

Through the process of describing the H-function in terms of contour integrals, integral identities, such as the Mellin-Barnes integral representation, provide a strong analytical tool. This representation not only makes it easier to compute the H-function in a variety of applications, but it also makes it possible to do transformations between multiple domains, such as moving from the time domain to the frequency domain whenever signal processing is being performed. These integral representations are particularly helpful in domains such as engineering and physics, where they are used to facilitate the assessment of solutions to differential systems and integral equations (Mcintosh, 2007).

RECENT DEVELOPMENTS IN HYPERGEOMETRIC SUMMATION

The addition of series having components that exhibit a hypergeometric structure is the subject of hypergeometric summation, a subfield of mathematics. In combinatorics, number theory, and special functions, the idea of hypergeometric sums has extensive applications that go beyond the standard binomial expansion. Mathematical physics, particularly in the study of quantum field theory and statistical mechanics, as well as algebraic geometry and modular forms have recently benefited from its use. This development is due to the interaction between computer methods, contemporary algebraic techniques, and our growing knowledge of the symmetry that underlies these sums (Purohit and Yadav, 2006).

In the context of binomial coefficients, hypergeometric sums with terms obtained from generalised hypergeometric functions emerge. The generalised hypergeometric series give an expression for these sums in a more generic form (Denis et al., 2006):

$$ {}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{(b_1)_n (b_2)_n \ldots (b_q)_n} \frac{z^n}{n!}, \quad 4 $$

where z is the series' argument and (a)n is the Pochhammer symbol for the increasing factorial. Numerous branches of mathematics, both theoretical and practical, make use of this series, which follows logically from the binomial series (Mori and Morita, 2016).

TRUNCATED BASIC HYPERGEOMETRIC FUNCTIONS

When compared to classical hypergeometric functions, basic hypergeometric functions are a generalisation of those functions. Several subfields of mathematics, such as combinatorics, number theory, and mathematical physics, are associated with the occurrence of these phenomena. The hypergeometric function, which is often referred to as rϕs, is defined by means of a series expansion. This expansion includes parameters, which are referred to as fundamental parameters, as well as variables that are raised to powers (Ono and Rolen, 2013).

An example of a variation is a truncated basic hypergeometric function, which is a variant in which the series is restricted to a finite number of terms. Because it enables simplifications and approximations that preserve crucial properties of the whole series, this truncation has significant consequences for both theoretical analysis and practical applications. However, these implications are not limited to the former (Mishra, 2013).

Definition and Notation

The basic hypergeometric function rϕs is defined as:

$$ {}_r\phi_s \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \frac{z^n}{(q;q)_n} \quad 5 $$

where (a;q)n denotes the q-Pochhammer symbol, defined as:

$$ (a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}) \quad 6 $$

For the truncated basic hypergeometric function, denoted as ${}_r\phi_s^{(N)}$. The series is limited to the first N terms:

$$ {}_r\phi_s^{(N)} \left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right) = \sum_{n=0}^{N} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \frac{z^n}{(q;q)_n} \quad 7 $$

Properties of Truncated Basic Hypergeometric Functions

Convergence

The fact that shortened fundamental hypergeometric functions converge is one of the most important characteristics of these functions. It is dependent on the parameters that are involved as well as the variable z whether or not the series will converge. The truncation ensures that the series will eventually converge to a finite value for N, and the error that is produced as a result of truncating the series can often be controlled and quantified (Gonzalez-Gaxiola and Santiago, 2012).

Special Cases

Under certain conditions, truncated basic hypergeometric functions may be reduced to more straightforward functions or to classical hypergeometric functions. By way of illustration, the production of classical outcomes may be achieved by making certain parameters equal or by allowing q to approach 1. In order to have a better grasp of the larger implications and uses of truncated functions, these specific situations are quite helpful (Denis et al., 2011).

EXISTING SUMMATION FORMULAE

A great number of significant summation formulas, especially those pertaining to fundamental hypergeometric series, have been developed as a result of the study of hypergeometric functions. As a result of their extensive structure and broad range of applications, these series are very important in a variety of subfields within the areas of mathematics and theoretical physics. Classical summation equations for fundamental hypergeometric functions will be discussed in this part. We will emphasise the significance of these formulae, as well as their derivation and provide examples to illustrate their use (Kexue and Jigen, 2011).

1. Basic Hypergeometric Series

A basic hypergeometric series, denoted as rϕs is defined by the series:

$$ {}_r\phi_s\left(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \frac{z^n}{(q;q)_n}, \quad 8 $$

where (a;q)n is the q-Pochhammer symbol defined as:

$$ (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad 9 $$

and z is a complex variable, while q is typically taken to be a positive real number less than one.

2. Classical Summation Formulae

One of the most celebrated summation formulae for basic hypergeometric series is the q-binomial theorem, which states:

$$ \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(qz;q)_\infty}. \quad 10 $$

This theorem provides a powerful identity connecting series with products, enabling computations in both combinatorial settings and analytic contexts.

Another essential formula is the **Bilateral Summation Formula** given by:

$$ \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n (c;q)_n} z^n = \frac{(c/a;q)_\infty}{(b;q)_\infty} \cdot \frac{1}{(az;q)_\infty}. \quad 11 $$

This formula extends the summation across all integers, providing broader applicability in contexts where negative indices are relevant.

3. Special Cases and Illustrations

To elucidate the above formulae, we can consider specific cases. For example, if we set a=b=1 in the q-binomial theorem, we obtain:

$$\sum_{n=0}^{\infty} \frac{(q;q)_n}{(q;q)_n} z^n = \frac{(z;q)_\infty}{(qz;q)_\infty};$$

12

which simplifies to a form that is useful in evaluating the convergence of series and products.

The Bilateral Summation Formula can also yield specific results. For instance, taking a=1, b=1, and c=1, we derive:

$$\sum_{n=-\infty}^{\infty} z^n = \frac{1}{1-z},$$

13

valid for $|z|<1$. This result is fundamental in the study of generating functions and in number theory, where generating functions play a pivotal role (Lovejoy, 2010).

4. Applications and Connections

Not only do these classical summation equations give beautiful identities, but they also make it easier to conduct more in-depth research enquiries into the characteristics of special functions and the applications of those functions. To provide one example, they play a significant role in the process of deriving identities that are associated with partition theory. In this theory, generating functions are used to count partitions of integers while adhering to certain constraints.

In the field of physics, specifically in the study of quantum mechanics and statistical mechanics, hypergeometric functions are found to emerge spontaneously in the solutions of a variety of differential equations. These equations include those that describe quantum harmonic oscillators and potential wells. Within the realm of physics, the summation equations provide scientists with the ability to represent wave functions and probabilities in closed forms, hence facilitating the analytical solutions of complicated systems (Laughlin, 2008).

CLASSICAL SUMMATION FORMULAE FOR BASIC HYPERGEOMETRIC FUNCTIONS

In the field of special functions and combinatorial identities, the basic hypergeometric series, which is represented by the notation rϕs_{r}phi_{s}rϕs, is a fundamental class of series. The traditional hypergeometric series is generalised by these series by the incorporation of fundamental parameters. These parameters are often related with q-calculus and have substantial consequences in a variety of mathematical areas, such as number theory, combinatorics, and mathematical physics. In this part, we will investigate several traditional summing equations for fundamental hypergeometric functions. We will provide both theoretical underpinnings and examples to illustrate our points (Ahmad, 2008).

1. Definition and Properties of Basic Hypergeometric Functions

A basic hypergeometric series is typically defined as follows:

$$ {}_r\phi_s\left(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_s; q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n \cdots (a_r;q)_n}{(b_1;q)_n (b_2;q)_n \cdots (b_s;q)_n} \frac{z^n}{(q;q)_n} \quad 14 $$

where (a;q)n is the q-Pochhammer symbol defined by:

$$ (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k) \quad 15 $$

And $(q;q)_n = (q;q)_n = (1-q)(1-q^2)\cdots(1-q^n)$

A number of qualities are shown by fundamental hypergeometric functions. These properties include transformation formulae, special cases, and relations to other special functions. When the q parameter becomes closer to 1, they reduce to classical hypergeometric functions, which is the common denominator between the two families of functions (Andrews and Warnaar, 2007).

2. Classical Summation Formulae

There are a number of traditional summation formulas for fundamental hypergeometric functions that have been well recorded. These formulae represent profound linkages between various series and provide essential identities. This section will focus on some of the most important formulas, including:

The q-Binomial Theorem

One of the foundational results in the theory of basic hypergeometric functions is the q-binomial theorem, which states:

$$ \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(q;q)_\infty (bz;q)_\infty} \quad 16 $$

for $|q|<1$. This theorem establishes a connection between basic hypergeometric series and binomial coefficients, serving as a powerful tool in combinatorial enumeration.

The Basic Hypergeometric Series Identity

Another classical summation formula involves the basic hypergeometric series itself:

$$ {}_{r+1}\phi_r\left(a_1, a_2, \ldots, a_r, q^{-n}; b_1, b_2, \ldots, b_s; q; 1\right) = \frac{(b_1, b_2, \ldots, b_s; q)_n}{(a_1, a_2, \ldots, a_r; q)_n} \quad 17 $$

In addition to demonstrating the intrinsic symmetry and structure that are present within these series, this identity offers a summation formula that makes the assessment of fundamental hypergeometric series at certain values much more straightforward (C.M. Joshi, 2005).

CONCLUSION

This study has derived new representation of the H function and showed that it has a great importance in the summation of hypergeometric series. The derived identity offers a new method that not only improves existing summation methods but also enriches the theoretical study of special functions. More precisely, application of this identity will lead to the search of more effective ways to manipulate with different hypergeometric series which may be useful in various branches of mathematical analysis, mathematical and theoretical physics and some branches of engineering. Even though the identity's application has shown a great deal of potential, its extension to more generalized forms of hypergeometric functions could be investigated in future studies. Moreover, the difficulties concerning the application of this identity to wider contexts and the possible drawbacks of its usage in higher-dimensional scenarios need to be studied. Finally, this work presents a useful addition to the study of special functions and reveals new approaches for the summation of hypergeometric series.

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