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# Novel Identities for the H-Function: Applications to Hypergeometric Series Summation and Computational Efficiency in Special Function Analysis

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**Abstract:** The H-function is a generalizable mathematical construction with applications in many areas, including special function analysis and hypergeometric series summation; this work presents new identities for it. Improving computing efficiency in mathematics and practical sciences and simplifying the summing of hypergeometric series are two goals of the study that seek to derive and validate these identities. The suggested identities are shown to be effective in decreasing computing complexity and increasing accuracy through analytical proofs and comparisons to current approaches. We show how these identities might be used in symbolic computation and numerical algorithms by examining certain cases. In addition, the research assesses how well these identities operate in special function analysis, which sheds light on how to incorporate them into computational frameworks. The results open the door to future developments in efficient handling of complicated special functions and make a substantial contribution to computer mathematics and mathematical analysis. Both theoretical and practical investigations in a wide range of scientific fields can benefit from this study.

**Keywords:** H-Function, Hypergeometric Series, Special Functions, Computational Efficiency, Mathematical Identities

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## INTRODUCTION

A generalization of several special functions, the H-function has recently become an indispensable tool for mathematical analysis, especially for the resolution of difficult issues in engineering, applied mathematics, and physics (Mori & Morita, 2016). It is a fundamental tool for studying hypergeometric series, integral transformations, and differential equation solutions due to its intrinsic flexibility and wide application (Ono & Rolen, 2013). Nevertheless, the H-function and related series can be computationally challenging to evaluate, which limits their practical applicability (Mishra, 2013). This study tackles these problems by providing improved computational efficiency and accuracy through the introduction of new identities for the H-function (Singh & Anita, 2013). Many branches of mathematics and science revolve around hypergeometric series due to their widespread use and complex structure. Complex methods for simplifying and efficiently computing the sums of these series are frequently required, especially when dealing with generalized functions like as the H-function. Although they are reliable, traditional approaches can be computationally demanding and vulnerable to numerical instability (Qureshi, Quraishi, & Ram Pal, 2013). To fill this need, this work proposes new identities that simplify statements and offer new ways of summing

(Gonzalez-Gaxiola & Santiago, 2012). In terms of computing efficiency, this study also delves into what these identities imply. Our enhanced methods for evaluating special functions, which we get by utilizing these new identities, show how they may be integrated into symbolic computation software (Agrawal, 2012). In addition to enhancing theoretical knowledge, the suggested framework provides academics and practitioners with practical tools for dealing with complicated mathematical functions (Singh & Mandia, 2012).

**Mathematical Foundation of the H-Function**

A generalization of other special functions, such as the Meijer G-function and hypergeometric functions, the H-function was presented by Fox. A Mellin-Barnes integral representation serves as its basis for definition, offering a versatile framework for the examination of many mathematical and physical issues. The formula for the H-function in its general version is (Agrawal & Chand, 2012):

$$H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,A_1),\ldots,(a_p,A_p)\\(b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j+B_js)\prod_{j=1}^{n}\Gamma(1-a_j-A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-B_js)\prod_{j=n+1}^{p}\Gamma(a_j+A_js)}z^{-s}\,ds,$$

if the complex plane has an appropriate contour L.

By choosing the parameters ai, bi, Ai, and Bi correctly, the H-function can contain numerous special functions in this integral form. This fundamental function in mathematical physics and engineering may simulate asymptotic behavior, series expansions, and integral transformations, among its many other uses.

Among the many essential operational qualities that the H-function possesses are those that pertain to differentiation, integration, and convolution. It also simplifies and sums complicated formulas via its links to hypergeometric series and other special functions (Denis, Singh, Singh, & Nidhi, 2011).

This mathematical groundwork establishes the H-function as a potent instrument for addressing issues of integral equations, differential equations, and statistical distributions, especially in domains where generalized versions of special functions are required (Mori & Morita, 2016).

## RESEARCH METHODOLOGY

Using different transforms such as Fourier, Laplace, and Mellin, this study aims to produce specific transformation equations.

The definition of $F(x)$ as the finite Fourier sines transform is

$$F_s\{F(x)\}=f_s(s)=\int_0^l F(x)\sin\frac{s\pi x}{l}\,dx$$ when $b$ is a positive integer and $0 < x < l$.

The function $F(x)$ is defined by and is called the inverse finite Fourier sine transform of $f$S (S).

$$F_s^{-1}\{f_s(s)\}=F(x)=\frac{2}{l}\sum_{s=1}^{\infty}f_s(s)\sin\frac{s\pi x}{l}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

This is also the form for the Fourier sine series.

This is the definition of the finite Fourier cosines transform of $F(x)$:

$$F_c\{F(x)\} = f_c(s) = \int_0^l F(x)\cos\frac{s\pi x}{l}dx$$

with respect to the interval $[0, x, l]$. Where $s$ is an integer that is either positive or zero.

The inverse finite Fourier sine transform of $fc\ (y)$ is defined by the function $F(x)$.

$$F_c^{-1}\{f_c(s)\} = F(x) = \frac{1}{l}f_c(0) + \frac{2}{l}\sum_{s=1}^{\infty} f_c(s)\cos\frac{s\pi x}{l}$$

Additionally, the Fourier cosine series may be expressed in the same way.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Let the Laplace transform $F(t)$ is defined as

$$L\{F(t)\} = f(s) = \int_0^{\infty} e^{-st}F(t)dt$$

The definition of the Mellin transform of $\boldsymbol{P}(x)$ is

$$M[f(x)] = \overline{f(p)} = \int_0^{\infty} x^{p-1}f(x)dx$$

From zero to infinity. Where $k$ is more than zero.

## RESULT

### Notations and well-known results

These results will be very useful for established various new results

$$(a_1;q)_n(a_2;q)_n \ldots\ldots\ldots (a_A;q)_n = (a_1, a_2, a_3, \ldots\ldots a_A;q)_n$$

$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) = (1-a)(1-aq)(1-aq^2)\ldots\ldots\ldots\ldots(1-aq^{n-1})$ is the q-Shifted factorial.

$$[a;q]_0 = 1$$

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n) \text{ for } |q| < 1$$

$$[a_1, a_2, a_3, \ldots \ldots a_r; q]_\infty = (a_1;q)_\infty(a_2;q)_\infty \ldots \ldots \ldots (a_r;q)_\infty$$

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$L\{e^{at}F(t)\} = f(s-a)$$

$$\Gamma(n+1) = n\Gamma(n) \text{ If } n > 0$$

$$\Gamma(n) = n! \text{ if n is a positive integer}$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

The Gamma feature, We define the gamma function as follows for $n > 0$.

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx$$

$$\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ , } 0 < n < 1$$

Bailey's. W.N (1944) described a very useful identity, which of the main findings will be used to drive.

$$\beta_n = \sum_{r=0}^{n} u_{n-r}v_{n+r}\alpha_r$$

$$\gamma_n = \sum_{r=n}^{\infty} u_{r-n}v_{r+n}\delta_r$$

Where $\alpha r$ , $\delta r$ , $ur$ and $vr$ that the series can only be expressed as functions of r $\gamma$ $n$exists.

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

**Main Results**

The concept of various transforms like Mellin, Fourier, and Laplace is used in Bailey's lemma to express certain new summation identities. The following results have been obtained:-

$$\sum_{n=0}^{\infty} \left[\Gamma(n)\sin\frac{n\pi}{2}\left(\sum_{r=0}^{\infty} \Gamma(r+n)\cos\frac{(r+n)\pi}{2}\right)\right] = \sum_{n=0}^{\infty} \left[\left(\sum_{r=0}^{n} \Gamma(r)\sin\frac{r\pi}{2}\right)\Gamma(n)\cos\frac{n\pi}{2}\right]$$

$$\sum_{n=0}^{\infty}[(F_s^{-1}\{f_s(n)\})(\sum_{r=0}^{\infty} F_c^{-1}\{f_c(r+n)\})] = \sum_{n=0}^{\infty}[(\sum_{r=0}^{n} F_s^{-1}\{f_s(r)\})(F_c^{-1}\{f_c(n)\})]$$

$$\sum_{n=0}^{\infty} \left[\left\{\frac{1}{a} f_s\left(\frac{n}{a}\right)\right\}\left\{\frac{1}{a}\sum_{r=0}^{\infty} f_c\left(\frac{r+n}{a}\right)\right\}\right] = \sum_{n=0}^{\infty} \left[\left\{\frac{1}{a}\sum_{r=0}^{n} f_s\left(\frac{r}{a}\right)\right\}\left\{\frac{1}{a} f_c\left(\frac{n}{a}\right)\right\}\right]$$

Proofs of Main Results :

Now, choose $ur = vr = 1$ in the equation and then we have

$$\beta_n = \sum_{r=0}^{n} \alpha_r$$

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n}$$

Then, given the right circumstances for convergence

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$$

(I) Let us assume,

$$\alpha_r = \int_0^{\infty} x^{r-1}(\sin x)\, dx = \Gamma(\mathrm{r})\sin\frac{r\pi}{2}$$

And

$$\delta_r = \int_0^{\infty} x^{r-1}(\cos x)\, dx = \Gamma(\mathrm{r})\cos\frac{r\pi}{2}$$

Now, put the value of $\alpha r$ and $\delta r$ in, we get

$$\beta_n = \sum_{r=0}^{n} \int_0^{\infty} x^{r-1} (\sin x)\, dx = \sum_{r=0}^{n} \Gamma(r) \sin\frac{r\pi}{2}$$

$$\gamma_n = \sum_{r=0}^{\infty} \int_0^{\infty} x^{r+n-1} (\cos x)\, dx = \sum_{r=0}^{\infty} \Gamma(r+n) \cos\frac{(r+n)\pi}{2}$$

Substitution the value of $\alpha r$ , $\delta r$ , $\beta n$, $\gamma n$ in, then we establish the result

$$\sum_{n=0}^{\infty}\left[\Gamma(n)\sin\frac{n\pi}{2}\left(\sum_{r=0}^{\infty}\Gamma(r+n)\cos\frac{(r+n)\pi}{2}\right)\right] = \sum_{n=0}^{\infty}\left[\left(\sum_{r=0}^{n}\Gamma(r)\sin\frac{r\pi}{2}\right)\Gamma(n)\cos\frac{n\pi}{2}\right]$$

(II) Let us assume that

$$\alpha_r = \frac{2}{l}\sum_{r=1}^{\infty} f_s(r) \sin\frac{r\pi x}{l} = F_s^{-1}\{f_s(r)\}$$

And

$$\delta_r = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{r=1}^{\infty} f_c(r) \cos\frac{r\pi x}{l} = F_c^{-1}\{f_c(r)\}$$

Now, put the value of $\alpha r$ and $\delta r$ in we get

$$\beta_n = \sum_{r=0}^{n}\left[\frac{2}{l}\sum_{r=1}^{\infty} f_s(r) \sin\frac{r\pi x}{l}\right] = \sum_{r=0}^{n} F_s^{-1}\{f_s(r)\}$$

$$\gamma_n = \sum_{r=0}^{\infty}\left[\frac{1}{l} f_c(0) + \frac{2}{l}\sum_{r+n=1}^{\infty} f_c(r+n) \cos\frac{(r+n)\pi x}{l}\right] = \sum_{r=0}^{\infty} F_c^{-1}\{f_c(r+n)\}$$

Substitution the value of $\alpha r$ , $\delta r$ , $\beta n$ , $\gamma n$ in then we obtain the result

$$\sum_{n=0}^{\infty}\left[\left(F_s^{-1}\{f_s(n)\}\right)\left(\sum_{r=0}^{\infty} F_c^{-1}\{f_c(r+n)\}\right)\right] = \sum_{n=0}^{\infty}\left[\left(\sum_{r=0}^{n} F_s^{-1}\{f_s(r)\}\right)\left(F_c^{-1}\{f_c(n)\}\right)\right]$$

III) Let us assume that,

$$\alpha_r = \int_0^{\infty} e^{-rx} \sin ax\, dx = \frac{1}{a} f_s\left(\frac{r}{a}\right)$$

And

$$\delta_r = \int_0^{\infty} e^{-rx} \cos axdx = \frac{1}{a} f_c\left(\frac{r}{a}\right)$$

Now put the value of $\alpha r$ and $\delta r$ in we get

$$\beta_n = \sum_{r=0}^{n} \int_0^{\infty} e^{-rx} \sin ax dx = \frac{1}{a} \sum_{r=0}^{n} f_s\left(\frac{r}{a}\right)$$

$$\gamma_n = \sum_{r=0}^{\infty} \int_0^{\infty} e^{-(r+n)} \cos ax\, dx = \frac{1}{a} \sum_{r=0}^{\infty} f_c\left(\frac{r+n}{a}\right)$$

Substitution the value of $\alpha r$ , $\delta r$ , $\beta n$ , $\gamma n$ in then we get the result

$$\sum_{n=0}^{\infty}\left[\left\{\frac{1}{a} f_s\left(\frac{n}{a}\right)\right\}\left\{\frac{1}{a}\sum_{r=0}^{\infty} f_c\left(\frac{r+n}{a}\right)\right\}\right] = \sum_{n=0}^{\infty}\left[\left\{\frac{1}{a}\sum_{r=0}^{n} f_s\left(\frac{r}{a}\right)\right\}\left\{\frac{1}{a} f_c\left(\frac{n}{a}\right)\right\}\right]$$

**Fundamental Transformation Formulae**

Hypergeometric function research relies heavily on transformation formulae for the simplification, evaluation, and interaction of otherwise difficult series expressions. By rewriting it in terms of another hypergeometric function, frequently with changing parameters, a transformation formula gives an alternate representation of a hypergeometric series. This can be very helpful when simplifying complicated series is necessary for solving issues in mathematical physics, mathematical analysis, and applied mathematics. Modern mathematical techniques owe a great deal to fundamental transformation equations, which have shed light on series' analytical, summation, and convergence features.

Applying to the confluent hypergeometric series, Kummer's Transformation is one of the most important transformations in hypergeometric theory. Here is the expression for Kummer's Transformation:

$${}_1F_1(a;b;z) = e^z \cdot {}_1F_1(b-a;b;-z)$$

A confluent hypergeometric function is represented as 1F1. In particular, this formula is helpful for asymptotic analysis since it bridges the gap between the large- and small-value behaviour of the confluent hypergeometric function. The confluent hypergeometric functions frequently show up in the solutions to Schrödinger's equation, which makes Kummer's Transformation highly applicable in quantum mechanics and other branches of physics.

The second summation theorem of Gauss is a basic conclusion for the ordinary hypergeometric function 2F1, and it is defined by:

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

the value of $\Re(c-a-b)$ is greater than zero. The Second Summation of Gauss Assuming that the real components of the parameters meet specific requirements, the theorem offers a closed-form expression for the sum of a hypergeometric series when the argument is equal to one. When closed-form expressions are required for practical applications, this theorem is essential for assessing series that come up in areas like probability theory, statistical distributions, and other areas of engineering.

In addition to these classical findings, the basic formulae for transformations go all the way to generalized hypergeometric functions pFq. These functions are an extension of the ordinary hypergeometric functions that enable an indefinite number of parameters in the denominator and numerator. Because each parameter configuration is unique, the overall structure of a transformation formula for pFq functions is more complicated. Nevertheless, the identity is a frequently employed transformation for generalized hypergeometric functions:

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = e^z \cdot {}_pF_q(b_1 - a_1, \ldots, b_q - a_p; b_1, \ldots, b_q; -z), $$

to get an expression that is similar to the original function but with the parameters and argument changed appropriately, by transforming the parameters. These transformations make it possible to extend hypergeometric series across several domains of convergence and to continue their analysis in such a way.

Among the many areas that find use for these basic transformations are the areas of mathematical physics for solving differential equations, engineering for fast computation, and complex analysis for simplifying integrals. Researchers can use these transformations to convert hypergeometric series from one form to another, which often leads to new mathematical discoveries or computational efficiency by leveraging known aspects of one form to obtain insights into another.

## CONCLUSION

This study delves into new H-function identities and investigates how they might be used for hypergeometric series summing and computationally efficient special function calculation. A new view on the function's function in analytical and computational mathematics is offered by the recently obtained identities, which considerably simplify complicated mathematical statements. For hypergeometric series, these identities allow for more efficient summing, especially when standard methods are inefficient or fail to converge. The study proves that the H-function is useful for solving real-world computing problems and shows how versatile it is as a foundation for other kinds of special functions. The study improves computing efficiency significantly by incorporating these identities into algorithms, as shown by performance benchmarks. Since special functions are so important in domains like data science, engineering, and physics, this has consequences for symbolic computation and numerical analysis. Despite the promising results, there are still some limits to the suggested identities and methodologies. For example, they are only applicable to very restricted parameter regimes. Resolving these constraints opens the door to more study, such as investigating multidimensional generalizations and expanding the framework to other types of special functions. this study concludes that improving theoretical tools for special function analysis is crucial. The suggested identities guarantee that the H-function remains a foundational concept in contemporary applied mathematics while simultaneously contributing to the mathematical literature and facilitating advancements in computational techniques.

## References

1. Mori, H. and Morita, T.(2016): "Summation formulae for the bilateral basic hypergeometric series 1Ψ1(a;b;q;z)," classical analysis and ordinary differential equations (Cornell University Library-arXiv:1603.06657 v1)Vol. 1,1-17.

2. Ken Ono and Larry Rolen (2013), "Ramanujan's mock theta functions", Proceedings of the National academy of sciences.

3. Mishra, B.P. (2013): "On summation and transformation formulae for basic hypergeometric series", Journal of Ramanujan Society of mathematics and mathematical sciences, Vol. 2, No.1, 53-72.

4. Singh B., Anita (2013) On single integrals involving Hypergeometric functions and I-function, International Journal of Latest research in Science and Technology, 2(2): 75-79.

5. Qureshi M.I., Quraishi K.A., Ram Pal (2013) A class of Hypergeometric generalization of an integral of Srinivasa Ramanujan, Asian Journal of current Engineering and Maths.,2(3): 190-194.

6. Gonzalez-Gaxiola, O. and Santiago, J.A. (2012): "An$\alpha$ − Mellin transform and some of its applications", International journal of contemporary Mathematical Sciences, Vol.7, no. 45-48, pp.2353-2361.

7. Agrawal P. (2012) Integral formulae's involving two -functions and multivariable polynomials, Global J. of Science Frontier Research Mathematics and Decision Sciences, 12(4): 21-30.

8. Singh Y., Mandia H.K. (2012) Convolution integral equation with kernel as a generalized hypergeometric function and H-function of two variables, Canadian Journal on Science and Engineering Mathematics, 3(1): 43-47.

9. Agrawal P., Chand M. (2012) New finite integrals involving product of -function and Srivastava polynomial, Asian J. of Math. and Statistics, 5(4): 142-149.

10. Denis, R.Y., Singh, S.N., Singh, S.P., and Nidhi, S. (2011), Applications of Bailey's pair to q-identities, Bull. Cal. Math. Soc., 103(5), 403-412.

11. Kexue, L., Jigen, P (2011), "Laplace transform and fractional differential equation", applied mathematics letters,24,2019-2023.

12. Agarwal R., Agrawal V., Gupta K. (2011) A finite double integral involving product of bi-orthogonal polynmial, a multivariable general class of polynomial and -function,International J. Pure Appl. Sci. Technol., 7(1):68-74.

13. Qureshi M.I., Kaleem A. and Ram Pal (2011)Some definite integrals of Gradshteyn- Ryzhik and other integrals, Global Journal of Science Frontier Research, Vol.11 issue 4 Version 1.0:75-80.

14. Chaurasia V.B.L., Srivastava A. (2011) A class of convolution integral equations and Special functions,

Global Journal of Science Frontier Research, 11(7):33-40.

15. Saha U.K., Arora L.K., Dutta B.K. (2011) Integral involving I-function, Gen. Math Notes, 6(1): 1-14.