Journal of Advances in Science and Technology
Vol. 22, Issue No. 2, April-2025, ISSN 2230-9659

DOI: https://doi.org/10.29070/a4zxsf49

Analytical study of the generalized I-function and its applications in statistical distributions

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Abstract: The "I-function" refers to a certain class of special functions known as the Generalized Fox H-function or an extension of Saxena's I-function. It is a versatile instrument used in many mathematics and technical domains, especially in communication systems and signal processing, often utilized to represent intricate phenomena. In statistics, complicated probability distributions are represented and analyzed using generalized I-functions, especially when conventional distributions are inadequate. The main of this paper is to discuss the Generalized I-Function and Its Applications in Statistical Distributions hence, In this work, we covered the fundamental idea and description of G-Functions, G-I-Functions, and Integrals Related to I-Functions. We continued by evaluating the Generalized I-Function's uses in statistical distributions in this article. It is concluded from the overall paper that the Generalized I-function provides statisticians with a powerful and adaptable toolset for analyzing complex data structures, improving model fit, and capturing intricate probabilistic behaviors.

Keywords: generalized, generalized functions, applications, statistical distribution, Saxena's I-function

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INTRODUCTION

The function developed by the first author represents the most generalized form of special functions, namely including generalized hypergeometric functions. This function developed during the resolution of dual integral equations in their general form. These equations emerge in fundamental form when addressing mixed boundary issues in potential theory, energy diffusion, and population dynamics. The I function may provide a comprehensive picture of the solutions to all such issues.

Because of their exceptional qualities, I function therefore broaden the scope of special functions that have been created and used in a number of disciplines, including physics, engineering, applied mathematics, astronomy, and combinatory. This book's primary goal is to serve as a platform for freshly published I function theories and formulae, along with any potential applicability to other fields of study. This book gives readers the chance to learn about current developments in this function and acquire the abilities required to use sophisticated mathematical methods to resolve challenging issues. The features of the I function and the many applications of mathematical analysis are often the subject matter.

The Mellin-Barnes type integral provides a generic representation of the I-function for a single variable:

$$I^{m,n}_{p,q}\left[z\,\middle|\,\begin{matrix}(a_i,\alpha_i,A_i)_{1,p}\\(b_j,\beta_j,B_j)_{1,q}\end{matrix}\right]=\frac{1}{2\pi i}\int_L \phi(s)z^{-s}\,ds \qquad (1)$$

This generalized function not only provides deep analytical insight but also supports practical applications in statistical modeling, fractional calculus, and mathematical physics, where traditional functions fail to capture the intricacies of the underlying phenomena.

GENERALIZED FUNCTION

A generalized function, referred to as a "distribution" or "ideal function," is identical to a given regular sequence and includes all regular sequences of functions that are very well-behaved. Simply said, a generalized function is a function that has had its concept expanded upon. According to scientific theory, the amount of force exerted on a baseball when hit by a bat changes with the passage of time. The force is not really a function as the model assumes that the bat's momentum transfers instantly. Actually, it's only a multiple of the delta function. Included in the set of distributions are Radon measures and locally integrable functions. If you think about statistical distributions, the term "distribution" immediately comes to mind.

Generalized functions may be defined as continuous linear functionals over a space of functions that are endlessly differentiable. Doing it this way ensures that all generalized functions have derivatives that are continuous. Among generalized functions, the delta function is often the one seen most frequently. A comprehensive and thorough account of the region is found in the multi-volume work by Gel'fand and Shilov (1964), whilst Vladimirov (1971) has a beautiful analysis of distributions from the perspective of a physicist. Both of these works are considered to be quite important. One of Schwarz's findings demonstrates that it is not possible to define distributions in a consistent manner across the complex numbers $\mathbb{C}$. .

Distributions may be added to one another; however, When distributions have simultaneous single support, multiplying them becomes impractical. This is because adding distributions is possible. Despite this, it is feasible to get a different distribution by taking the derivative of the distribution that you are currently using. Because of this, They could just solve a partial differential equation that is linear in nature. In such instance, we say that the distribution is a weak solution. For instance, one may reasonably ask about the solutions u of Poisson's equation given any locally integrable function f.

$$\nabla^2 u = f \quad (2)$$

by requiring that the equation be true when applied to distributions, that is, when the two sides reflect the same distribution. Any distribution p(x) may have its derivatives defined by:

$$\int_{-\infty}^{\infty} p'(x) f(x)\, dx = -\int_{-\infty}^{\infty} p(x) f'(x)\, dx \quad (3)$$

$$\int_{-\infty}^{\infty} p^{(n)}(x) f(x)\, dx = (-1)^n \int_{-\infty}^{\infty} p(x) f^{(n)}(x)\, dx. \quad (4)$$

Distributions vary from functions due to their covariant nature, meaning they exhibit a push-forward behavior. Considering a differentiable function $a: \Omega_1 \to \Omega_2,$ a distribution $T \; on \; \Omega_1$ allows for the deployment of a distribution on $\Omega_2,$ An actual function, on the other hand, f on $\mathrm{f \; on} \; \Omega_2$ performs a function on the rear of the Ω_1 $namely\ f\ (a)x))$.

As a matter of fact, distributions are defined as the topological twin of the smooth functions of compact support. As an example, the linear functional is represented by the delta function $\delta(f) - f(0)$. A function g's associated distribution is

$$T_g(f) = \int_\Omega f\, g, \quad (5)$$

According to the measure μ, the distribution that corresponds to it is:

$$T_\mu(f) = \int_\Omega f\, d\mu. \quad (6)$$

In the event that you possess a distribution T and a smooth map α, the pushforward $\alpha * T$ is defined as the following:

$$\alpha_* T(f) = T(f \circ \alpha), \quad (7)$$

and the description of T's derivative $D\,T(f) = T(D^\dagger f)$ where $D^\dagger$ is a formal adjoint of the letter D. Example: the delta function's first derivative may be written as:

$$\frac{d}{dx}[\delta(f)] = -\left.\frac{df}{dx}\right|_{x=0}. \quad (8)$$

For each function space, the topology is the factor that defines whether linear functionals are continuous, or are in the dual vector space. This is true regardless of the function space. When it comes to topology, the family of seminorms is what defines it,

$$N_{K,\alpha}(f) = \sup_K \|D^\alpha f\|, \quad (9)$$

the supremum is denoted by the symbol sup. With regard to compact subsets, it is in agreement with the C-infty topology. There is a convergence of a sequence in this topology, $f_n \to f, iff$ a compact set K exists in such a way that includes all f_n possess support in K as well as in every derivative $D^a f_n$ uniformly converges to is the $D^a\ f\ in\ K$. Since this is the case, the constant function 1 may be classified as a distribution $f_n \to f\ then$

$$T_1(f_n) = \int_K f_n \to \int_K f = T_1(f). \quad (10)$$

THE GENERALIZED I-FUNCTION

In the year 1982, Saxena presented the I-function, which is an extension of Fox's H-function. Saxena first presented the I-function. The I-function of a single variable was first established by Saxena (1982), and further research on the topic was conducted by Kumbhat and Khan (2001), Sharma and Ahmad (1992), Sharma and Srivastava (1992), Sharma and Tiwari (1993), Vaishya, Jain, and Verma (1989), and other researchers.

A strong mathematical tool that generalizes a number of other special functions, the generalized I-function is crucial for statistical distribution analysis and building.

According to Saxena et al., the Generalized I-function is an ultra-generalized function that includes a broad family of special functions, including the Wright function, Fox H-function, and Meijer G-function. It was built to model generalized statistical distributions, solve fractional differential equations, and perform complex integral transformations. When heavier tails, skewed behavior, or multi-modal behavior in real-world events cannot be described by simpler functions, this function is very helpful.

By adding more parameters and permitting a more intricate structure in the gamma functions that appear in its integral form, the I-function expands upon these. Because of this improvement, the I-function may now be used to describe a wider variety of generalized functions, particularly in applied statistical distributions and intricate integral assessments.

The following is how the I-function of a single variable was defined:

$$I[y] = I^{m,n}_{p_i,q_i;r}[y] = I^{m,n}_{p_i,q_i;r}\left[y\left|\begin{matrix}[(a_j,\alpha_j)_{1,n}],[(a_{ji},\alpha_{ji})_{1+n,p_i}]\\ [(b_j,\beta_j)_{1,m}],[(b_{ji},\beta_{ji})_{1+m,q_i}]\end{matrix}\right.\right]$$
$$= \frac{1}{2\pi i}\int_L \theta(s)y^s ds, \qquad (11)$$

Where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left[\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ij}s)\prod_{j=1+n}^{p_i}\Gamma(a_{ij}-\alpha_{ij}s)\right]} \qquad (12)$$

Let p_i (i = 1, 2, …, r), q_i (i = 1, 2, …, r), m, and n be integers such that $0 \le n \le p_i$ and $0 \le m \le q_i$ (i = 1, 2, …, r). The variables r_i are finite, while αj, β_j, α_{ij}, and β_{ij} are real and positive. The variables a_j, b_j, a_{ij}, and b_{ji} are complex numbers, ensuring that $\alpha_j(b_h + v) \neq \beta_h(a_j - 1 - k)$ for v, k = 0, 1, 2, …, h = 1, 2, …, m, and j = 1, 2, …, n. L denotes a contour extending from σ − i∞ to σ + i∞ (where σ is real) in the complex s-plane, encompassing the specified points.

$$s = \frac{(a_j - 1 - k)}{a_j}; j = 1,2,..,n, v - 0,1,2,..,$$

$$s = \frac{(b_h + v)}{\beta_j}; j = 1,2,..,m, v - 0,1,2,..,$$

represents the left and right sides of L, correspondingly.

For r=1, the I-function reduces to the H-function.

Note 4. Putting r = 1 and $\tau_1 = \tau_2 = \ldots = \tau_3 = 1$, then ℵ- function simplifies to the established H-function as defined by . Fox.

$$H_{P,Q}^{M,N}[Z] = H_{P,Q}^{M,N}\left[Z\left|\begin{matrix}(a_j,\alpha_j)_{1,P}\\(b_j,\beta_j)_{1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \Omega(s) Z^s ds \tag{13}$$

Where $i = \sqrt{-1}\ Z \neq 0$ *and*

$$\Omega(s) = \frac{\prod_{j=1}^{M}\Gamma(b_j-\beta_j s)\prod_{j=1}^{N}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j s)\prod_{j=M+1}^{Q}\Gamma(1-b_j+\beta_j s)} \tag{14}$$

Saxena's definition of the I-function, which is more generic than the H-function of Fox, is accomplished by the use of the Mellin-Barnes type contour integral that is shown below:

$$I[z] = I_{p_i,q_i;r}^{m,n}[z] = I_{p_i,q_i;r}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{15}$$

Where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}s)\right\}}. \tag{16}$$

Definition of the Generalized I-Function

In 1982, Saxena created a new function known as the I function or Saxena's I function, which was inspired by the significance of Fox's H and G functions in mathematical analysis. The I function has the following definition:

$$I(z) = I_{p_i,q_i;r}^{m,n}\left[x\left|\begin{matrix}(a_j,a_j)_{j,n};(a_{ji},\alpha_{ji})_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(a_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] \tag{17}$$

$$= \frac{1}{2\pi i}\int_L \phi(\xi) x^{\xi} d\xi$$

$$\phi(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n+1}^{p_i}\Gamma(a_{ji}-\alpha_{ji}\xi)\right\}} \tag{18}$$

The contour L extends from σ - i ∞ to σ + i ∞, ensuring that all the poles of $\Gamma(b_j-\beta_j\xi), j = 1,2,\ldots.n,$ are positioned to the right, along with all the poles of $\Gamma(1-a_j-a_j\xi), j = 1,2,\ldots.m,$ In the direction to the left of L, the integral will converge if $p+q<2<(m+n), |\arg x|<(m+n-\frac{p}{2}-\frac{q}{2})\pi$

On the other hand, L is a loop that begins and ends at σ + i∞, and it encompasses all of the poles of $\Gamma(1-a_j-a_j\xi), j=1,2,\dots.m$ a single time in the negative direction, yet none of the poles of $\Gamma(b_j-\beta_j\xi), j=1,2,\dots.n,$ the integral converges of $q\geq 1$ and $p<q$ or $p=q$ and $|x|<1$

L is a loop that begins and ends at σ − i∞ and encircles all of the poles of the set of symbols $\Gamma(1-a_j-a_j\xi), j=1,2,\dots.m,$ once in the affirmative direction, although none of the poles of, $\Gamma(b_j-\beta_j\xi), j=1,2,\dots.n,$ t the integral converges of p ≥ 1 and either p > q or p = q and x < 1.

Under the circumstances that are now in place, this function is implemented:

a) $\lambda > 0, |arg\ x| < \pi\frac{\lambda}{2}$
b) $\lambda > 0, Re\ (\mu+1) < 0, |arg\ x| > \pi_\lambda/2$

λ and μ indicate the amounts

$$\lambda=\sum_{j=1}^{p}\alpha_j+\sum_{j=1}^{q}\beta_j=\max_{1\leq i\leq r}\left[\sum_{j=q+1}^{s_i}\alpha_{ji}+\sum_{j=q+1}^{t_i}\beta_{ji}\right] \quad (19)$$

$$\mu=\sum_{j=1}^{m}(b_j-\beta_j\sigma)-\sum_{j=1}^{n}(a_j-\alpha_j\sigma)$$

$$-\max_{1\leq i\leq r}\left[\sum_{j=n+1}^{p_i}(a_j-\alpha_j\sigma)-\sum_{j=m+1}^{q_i}(b_j-\beta_j\sigma)+\frac{p_i}{2}-\frac{q_i}{2}\right] \quad (20)$$

Here,

$\xi=\sigma+it$

The Kernel Function $\phi(s)$

In the realm of complex-valued functions, the integrand kernel ϕ(s) is a function that is produced by the products of gamma functions that possess positive real powers. It may be described as:

$$\phi(s)=\frac{\prod_{j=1}^{m}\Gamma^{B_j}(b_j+\beta_j s)\cdot\prod_{i=1}^{n}\Gamma^{A_i}(1-a_i-\alpha_i s)}{\prod_{j=m+1}^{q}\Gamma^{B_j}(1-b_j-\beta_j s)\cdot\prod_{i=n+1}^{p}\Gamma^{A_i}(a_i+\alpha_i s)}, \quad (21)$$

Where $\Gamma^{B_j}(x)=[\Gamma(x)]^{B_j}$ denotes the gamma function elevated to a positive exponent B_j While the denominator is responsible for determining the zeros that correspond to the poles of the integrand, the numerator is responsible for introducing the poles of the integrand. This allows for control over the pole position, spacing, and multiplicity via the parameters $a_i, b_j, \alpha_i, \beta_j, A_i, B_j$

The Contour of Integration

In order to run from, the contour L is selected c−i∞, where to the extent that it distinguishes between the poles of the gamma functions that are found in the numerator and those that are found in the denominator. In the complex plane, this is often a Bromwich route that is vertical. Because of the selection of ccc, convergence of the integral is guaranteed, and crossing singularities of the integrand are avoided if possible.

Convergence Conditions

Depending on the parameters and the asymptotic behavior of the gamma functions that are involved, the integral will gradually converge under specific circumstances. In order for the function to converge as |z|→0, it is necessary to have a valid condition

$$\sum_{j=1}^{q} \beta_j B_j - \sum_{i=1}^{p} \alpha_i A_i > 0. \tag{22}$$

By contrast, in the case of convergence as |z|→∞, It is the opposite of the inequality. The presence of these requirements guarantees that the integrand will decay to an adequate degree at infinity and that the contour will not cross the poles.

Special Cases

There are two well-known special functions that are generalized by the I-function. It is possible to achieve the Meijer G-function when all of the scaling and power parameters are equal to one:

$$G_{p,q}^{m,n}(z) = I_{p,q}^{m,n}\left[z \mid (a_i, 1, 1), (b_j, 1, 1)\right] \tag{23}$$

In situations when the multiplicity parameters are equal to one, but scaling is maintained, the Fox H-function is established:

$$H_{p,q}^{m,n}(z) = I_{p,q}^{m,n}\left[z \mid (a_i, \alpha_i, 1), (b_j, \beta_j, 1)\right]. \tag{24}$$

Therefore, the I-function is an extension of the Fox H-function since it allows for the possibility of higher-order poles with the addition of the A_i, B_j parameters.

The Generalized I-function has significant versatility, making it suitable for modeling situations where traditional functions fall short. It is especially efficacious in scenarios characterized by power-law behavior, skewed or heavy-tailed statistical distributions, and fractional-order differential equations. Applications include fractional calculus, quantum statistics, astronomy, and signal processing, whereby the ability to manipulate scaling and pole multiplicity offers a significant advantage in elucidating complicated processes.

INTEGRALS PERTAINING TO I-FUNCTIONS

Integrals involving product of the I- function

An application of Fox-Wright's Generalized Hypergeometric Function and the I-function product was used to construct specific integrals in this study. These integrals, which are unified and universal in character, provide a number of known and novel findings as specific examples. Some unusual examples are also documented for illustrative purposes. In this study, we employed Saxena's I-function, which is defined as:

$$I^{m,n}_{p_i,q_i:r}[z] = I^{m,n}_{p_i,q_i:r}\left[z\left|\begin{matrix}(a_{j'},\alpha_{j'})_{1,n};(a_{j'i'},\alpha_{j'i'})_{n+1,p_i}\\(b_{j'},\beta_{j'})_{1,m};(b_{j'i'},\beta_{j'i'})_{m+1,q_i}\end{matrix}\right.\right]$$

$$= \frac{1}{2\pi\omega}\int_L \varphi(\xi)\, z^{\xi}\, d\xi \qquad (25)$$

Where

$$\varphi(\xi) = \frac{\prod_{j'=1}^{m}\Gamma(b_{j'}-\beta_{j'}\xi)\prod_{j'=1}^{n}\Gamma(1-a_{j'}+\alpha_{j'}\xi)}{\sum_{i'=1}^{r}\left\{\prod_{j'=m+1}^{q_i}\Gamma(1-b_{j'i'}-\beta_{j'i'}\xi)\prod_{j'=n+1}^{p_i}\Gamma(a_{j'i'}-\alpha_{j'i'}\xi)\right\}} \qquad (26)$$

and =ω −1. With regard to the requirements that are placed on the I-function's various parameters

The Generalized hypergeometric function of Wright ${}_p\psi_q$ appearing is defined as follows in this paper:

$${}_p\psi_q\left[\begin{matrix}(e_1,\gamma_1),\ldots,(e_p,\gamma_p);\\(f_1,\delta_1),\ldots,(f_q,\delta_q);\end{matrix}\; x\right] = {}_p\psi_q\left[\begin{matrix}(e_j,\gamma_j)_{1,p};\\(f_j,\delta_j)_{1,q};\end{matrix}\; x\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(e_j+\gamma_j k)}{\prod_{j=1}^{q}\Gamma(f_j+\delta_j k)}\frac{x^k}{k!} \qquad (27)$$

where γ_i and δ_j (i = 1, …, p ; j = 1, …, q) which are genuine and optimistic, and

$$1+\sum_{j=1}^{q}\delta_j-\sum_{j=1}^{p}\gamma_j>0.$$

Additionally, we used the following results to ascertain the integrals:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n-k) \tag{28}$$

Main Results

In this section, we studied integrals using the exponential function, Wright's Generalized hypergeometric function, and the product of the I-function.

1st Integral

$$I_1 = \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} \; {}_p\psi_q \left[\begin{matrix} (e_j,\gamma_j)_{1,p}; \\ (f_j,\delta_j)_{1,q}; \end{matrix} \; ax^{\zeta}(t-x)^{\eta} \right]$$

$$\times \; I^{m,n}_{p_i,q_i:r} \left[yx^{\mu}(t-x)^{\nu} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right. \right] dx$$

$$= e^{-zt} t^{\rho+\sigma-1} \sum_{u=0}^{\infty} \sum_{k=0}^{u} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m,n+2}_{p_i+2,q_i+1:r} \left\{ yt^{\mu+\nu} \left| \begin{matrix} (1-\rho-\zeta k,\mu),(1-\sigma-(n-1)k-u,\nu)(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i},(1-\rho-\sigma-(\zeta+\eta-1)k-u,\mu+\nu) \end{matrix} \right. \right\} \tag{29}$$

Where

$$f(k) = \frac{\prod_{j=1}^{p} \Gamma(e_j+\gamma_j k)}{\prod_{j=1}^{q} \Gamma(f_j+\delta_j k)} \frac{a^k}{k!} \tag{30}$$

provided

i. $\mu \geq ,0\ \nu \geq 0$ (Not both zero simultaneously)

ii. ζ and η are numbers that are not negative and that means $\zeta\ \eta+ \geq 1$

iii. $A_i > 0$, $B_i < 0$; $|\arg y| < \frac{1}{2} A_i\pi$, $\forall$ $i \in 1,\ldots,r$;

Where

$$A_i = \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p_i} \alpha_{ji} + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q_i} \beta_{ji},$$

$$B_i = \frac{1}{2}(p_i - q_i) + \sum_{j=1}^{q_i} b_{ji} - \sum_{j=1}^{p_i} a_{ji},$$

And

$$\mathrm{Re}(\rho) + \mu \min_{1\leq j\leq m}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right] > 0, \quad \mathrm{Re}(\sigma) + \nu \min_{1\leq j\leq m}\left[\mathrm{Re}\left(b_j/\beta_j\right)\right] > 0.$$

Proof

$$I_1 = e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{(t-x)z}\ {}_p\psi_q\left[\begin{matrix}(e_j,\gamma_j)_{1,p};\\ (f_j,\delta_j)_{1,q};\end{matrix}\ ax^{\zeta}(t-x)^{\eta}\right]$$

$$\times\ I^{m,n}_{p_i,q_i:r}\left[y\,x^{\mu}(t-x)^{\nu}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p_i}\\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right]dx$$

At this time, replacing $e^{(t-x)z}$ by $\sum_{u=0}^{\infty} \frac{(t-x)^u}{u!} z^u$ in addition, by using the equations (25) and (27), we are able to:

$$I_1 = e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\frac{(t-x)^u}{u!}z^u\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(e_j+\gamma_j k)}{\prod_{j=1}^{q}\Gamma(f_j+\delta_j k)}\frac{a^k x^{\zeta k}(t-x)^{\eta k}}{k!}$$

$$\times\frac{1}{2\pi i}\int_L \varphi(\xi)\,y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}\,d\xi\,dx$$

$$= e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(e_j+\gamma_j k)}{\prod_{j=1}^{q}\Gamma(f_j+\delta_j k)}\frac{a^k x^{\zeta k}(t-x)^{\eta k+u}}{k!}\frac{z^u}{u!}$$

$$\times\frac{1}{2\pi i}\int_L \varphi(\xi)\,y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}\,d\xi\,dx$$

in light of the fact that (28), turns to

$$I_1 = e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u}\frac{\prod_{j=1}^{p}\Gamma(e_j+\gamma_j k)}{\prod_{j=1}^{q}\Gamma(f_j+\delta_j k)}\times\frac{a^k x^{\zeta k}(t-x)^{\eta k+u-k}}{k!}\times\frac{z^{u-k}}{(u-k)!}$$

$$\times\frac{1}{2\pi i}\int_L \varphi(\xi)y^{\xi}x^{\mu\xi}(t-x)^{\nu\xi}d\xi dx$$

Changing the sequence in which integration and summation are performed, we get the following:

$$I_1 = e^{-zt}\sum_{u=0}^{\infty}\sum_{k=0}^{u}f(k)\frac{z^{u-k}}{(u-k)!}\times\frac{1}{2\pi i}\int_L \varphi(\xi)y^{\xi}\left\{\int_0^t x^{\rho+\zeta k+\mu\xi-1}(t-x)^{\sigma+(\eta-1)k+u+\nu\xi-1}dx\right\}d\xi$$

f (k) is presented by the equation (30). In addition, by inserting x= t_s in the inner x- integral, the formula that was shown before may be reduced to:

$$I_1 = e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\gamma+\eta-1)k+u}\times\frac{1}{2\pi i}\int_L \varphi(\xi)y^{\xi}\left\{\int_0^1 s^{\rho+\zeta k+\mu\xi-1}(1-s)^{\sigma+(\eta-1)k+u+\nu\xi-1}ds\right\}d\xi$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u}f(k)\frac{z^{u-k}}{(u-k)!}t^{(\gamma+\eta-1)k+u}\times\frac{1}{2\pi i}\int_L \varphi(\xi)\frac{\Gamma(\rho+\zeta k+\mu\xi)\Gamma(\sigma+(\eta-1)k+u+\nu\xi)}{\Gamma(\rho+\sigma+(\zeta+\eta-1)k+u+(\mu+\nu)\xi)}y^{\xi}t^{(\mu+\nu)\xi}d\xi$$

Due to the fact that (25) is true, we have finally arrived at the intended outcome.

2nd Integral

$$I_2 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} \; {}_p\psi_q\left[\begin{matrix}(e_j,\gamma_j)_{1,p};\\(f_j,\delta_j)_{1,q};\end{matrix}\; ax^{\zeta}(t-x)^{\eta}\right]$$

$$\times\; I^{m,n}_{p_i,q_i:r}\left[yx^{-\mu}(t-x)^{-v}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] dx$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m,n+2}_{p_i+2,q_i+1:r}\left\{yt^{-\mu-v}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i},(\rho+\sigma+(\zeta+n-1)k+u,\mu+v)\\(\rho+\zeta k,\mu),(\sigma+(\eta-1)k+u,v),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right\} \quad (31)$$

Provided $\mathrm{Re}(\rho)-\mu \max\limits_{1\le j\le n}[\mathrm{Re}(\frac{a_j-1}{\alpha_j})]>0,\ \mathrm{Re}(\sigma)-v \max\limits_{1\le j\le n}[\mathrm{Re}(\frac{a_j-1}{\alpha_j})]>0,$

In addition to the sets of criteria (i) to (iii) that are provided with I1, the equation for f (k) is the following: (30).

Proof: It might be shown using lines that are comparable with those shown in (29).

3rd Integral

$$I_4 \equiv \int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-xz} \; {}_p\psi_q\left[\begin{matrix}(e_j,\gamma_j)_{1,p};\\(f_j,\delta_j)_{1,q};\end{matrix}\; ax^{\zeta}(t-x)^{\eta}\right]$$

$$\times I^{m,n}_{p_i,q_i:r}\left[yx^{-\mu}(t-x)^{v}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};(a_j,\alpha_j)_{n+1,p_i}\\(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right] dx$$

$$= e^{-zt}t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{u} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\zeta+\eta-1)k+u}$$

$$\times I^{m+1,n+1}_{p_i+2,q_i+1:r}\left\{yt^{-\mu+\nu}\left|\begin{matrix}(1-\sigma-(\eta-1)k-u,\nu),(a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i},(\rho+\sigma+(\zeta+\eta-1)k+u,\mu-\nu)\\(\rho+\zeta k,\mu),(b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i}\end{matrix}\right.\right\} \quad (32)$$

Provided $\mathrm{Re}(\rho)+\mu\min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)]>0,\ \mathrm{Re}(\sigma)-\nu\max_{1\le j\le n}[\mathrm{Re}(\frac{a_j-1}{\alpha_j})]>0,$ In addition to the sets of criteria (i) to (iii) that are provided with I1, the equation for f (k) is supplied by (30).

Proof: It might be shown using lines that are comparable with those shown in (29).

APPLICATIONS OF THE GENERALIZED I-FUNCTION IN STATISTICAL DISTRIBUTIONS

An integral of the Mellin-Barnes type is used to generate the generalized I-function, which has recently become an essential tool for building and analyzing complex statistical distributions. The system's structural generalizability allows it to include many famous special functions as special examples, including the Meijer G-function and the Fox H-function. This additional generality is a result of the incorporation of a third set of parameters into the Mellin–Barnes kernel. These parameters determine the order (or power) of the gamma functions. As a consequence of this, the I-function makes it possible to represent statistical processes that are not capable of being captured by traditional probability distributions.

General Form of I-Function Distributions

A probability density function (PDF) derived from the I-function, sometimes referred to as an I-distribution, is formulated by including the generalized I-function into the framework of the PDF. The standard representation of such a density is:

$$f(x)=k\cdot I^{m,n}_{p,q}\left[ax^b\left|\begin{matrix}(a_i,\alpha_i,A_i)_{1,p}\\(b_j,\beta_j,B_j)_{1,q}\end{matrix}\right.\right],\quad x>0, \quad (33)$$

where k is a constant that is used for normalizing, a and b are constants that are used for positive scaling and shaping, and the I-function $I^{m,n}_{p,q}$ is defined by use of a Mellin–Barnes integral that includes gamma functions that have parameters (a_i,α_i,A_i) and (b_j,β_j,B_j). This generic structure lets you fine-tune the distribution's location, scale, skewness, and tail behavior, giving it a strong base for probabilistic modeling.

I-Weibull Distributions for Lifetime Analysis

A significant use of the generalized I-function is the formulation of the I-Weibull distribution, which broadens the traditional Weibull distribution. The Weibull distribution is a recognized model in reliability theory and survival analysis; nonetheless, it presumes a monotonic hazard rate, which is limiting in several real-world scenarios. The I-Weibull distribution substitutes the exponential kernel in the Weibull density with the I-function, resulting in a much more adaptable framework. One way to represent the I-Weibull distribution's probability density function is as:

$$f(x) = kx^{\gamma-1} I_{p,q}^{m,n}\left[\lambda x^{\delta} \middle| \begin{array}{l} (a_i, \alpha_i, A_i)_{1,p} \\ (b_j, \beta_j, B_j)_{1,q} \end{array}\right], \quad x > 0, \tag{34}$$

where γ and δ regulate the size and form, and λ adds another scaling factor. In accordance with the application, this distribution may handle rising, falling, bathtub-shaped, or even oscillatory hazard rates thanks to the flexibility provided by the I-function component.

Generalized I-Gamma Distributions

Additionally, the generalized I-function makes it easier to extend the gamma distribution, resulting in the so-called I-Gamma distribution. Because of its conjugacy and tractability, the gamma distribution is essential to Bayesian statistics, queuing models, and reliability theory. Its capacity to simulate multimodal, skewed, or heavy-tailed events is constrained, nevertheless. By introducing an extra component via the I-function, an I-Gamma density function permits a broad range of distribution shapes. This density might look like:

$$f(x) = kx^{\mu-1} e^{-\theta x} I_{p,q}^{m,n}\left[a x^{r} \middle| \begin{array}{l} (a_i, \alpha_i, A_i)_{1,p} \\ (b_j, \beta_j, B_j)_{1,q} \end{array}\right], \quad x > 0, \tag{35}$$

where μ and θ control the density's traditional gamma-like component, while the I-function adds fractional or higher-order behaviors. When modelling biological or mechanical failure data, when ordinary gamma models are inadequate, this expanded version is very helpful.

Moment Analysis Using Mellin Transforms

I-function-based distributions have a number of advantageous characteristics, one of which is the availability of explicit formulations for moments via the use of Mellin transforms. In the event that the PDF is defined in the form $f(x) = kx^{\rho-1} I_{p,q}^{m,n}[ax^b | \cdots]$. Therefore, the s-th instant $\mathbb{E}[X^s]$ is calculated as follows:

$$\mathbb{E}[X^s] = \int_0^{\infty} x^s f(x) dx = k \cdot a^{-s/b} \cdot M_I(s), \tag{36}$$

where $M_I(s)$ consists of the Mellin transform applied to the I-function component of the density. The convergence of this integral is a function of the parameters (a_i, α_i, A_i) and (b_j, β_j, B_j). in addition to the contour that was selected for the Mellin–Barnes integral. Within the realm of statistical inference and parameter estimation, the analytical tractability of moments is very advantageous.

Bayesian Models and Priors Involving the I-Function

The prior and posterior distributions in Bayesian analysis often need to be selected in order to accurately represent the complicated domain knowledge or empirical behaviors that are being considered. The I-function architecture makes it possible to design non-standard priors that have tail behaviors that may be controlled and a plurality of modes. One such example of a prior distribution on a scale parameter θ is the selection of the following:

$$\pi(\theta) \propto I_{p,q}^{m,n}\left[\theta^{\alpha} \mid \cdots\right], \quad \theta > 0, \tag{37}$$

where it is possible to change the values of the I-function in order to generate heavier or lighter tails, greater or weaker concentration around a mode, or even single behaviors close to the origin. These priors are helpful in sparse Bayesian learning, hierarchical modelling, and nonparametric Bayes, which are all situations in which ordinary gamma or inverse gamma priors are neither sufficient nor enough.

Modelling Heavy-Tailed and Skewed Phenomena

The I-function distributions' ability to encapsulate large tails and skewness in empirical data is one of its most significant attributes. Numerous real-world events, including financial returns, Internet traffic, biological measures, and environmental factors, deviate from symmetric, light-tailed distributions such as the normal distribution. The adaptability of the I-function in delineating the characteristics of its integrand through α_i, β_j (which control scaling) and A_i, B_j which dictates pole multiplicity enables the modeling of a diverse array of skewed and heavy-tailed phenomena. Furthermore, imbalance in parameters may result in significantly distorted distributions, while escalating values of capable of producing fat tails of decay that is either polynomial or exponential.

Entropic and Thermodynamic Applications

In extended statistical mechanics, especially for non-extensive entropy measures like Tsallis, Rényi, or Sharma–Mittal entropy, the probability densities that optimize these entropies often include non-standard functions. It has been shown that, given appropriate restrictions, the resultant maximum entropy distributions may be articulated using the I-function. These distributions characterize physical systems exhibiting long-range interactions, memory effects, or multifractal features. In certain instances, the I-function-based PDFs assume the form:

$$f(x) = \frac{1}{Z} I_{p,q}^{m,n}\left[\eta x^{\theta} \mid \cdots\right], \tag{38}$$

where the parameters are selected to meet imposed limits like fixed energy or variance, and Z is the normalizing constant (partition function). The probabilistic description of complicated systems relies heavily on these I-function representations.

Compound and Mixture Distributions in Risk Models

In actuarial science, finance, and risk modeling, compound distributions often arise, when a random variable represents the summation of a random quantity of independent and identically distributed elements. If the summands adhere to an I-distribution and the quantity of summands conforms to a distribution like Poisson or geometric, the resultant compound distribution often maintains an analytically manageable I-function structure. These models are especially beneficial for assessing total claim amounts in insurance or total losses in financial portfolios. The convolution characteristics of the I-function, obtained from the

Mellin–Barnes integral representation, facilitate the computation of the compound distribution either precisely or using saddle point approximations.

Models for collective losses in actuarial and risk theory often include compound distributions, in which the entire loss $S = \sum_{i=1}^{N} X_i$ equals the total of N independent, normally distributed variables X_i. The aggregate distribution that results from combining claim amounts Xi that follow a generalized I-function distribution with claim counts N that follow a known discrete distribution (e.g., Poisson) may nevertheless have the shape of the I-function.

Let us assume that the density of each X_i is represented by:

$$f_X(x) = c \cdot x^{\rho-1} I_{p,q}^{m,n} \left[\alpha x^{\beta} \middle| \begin{matrix} (a_i, \alpha_i, A_i) \\ (b_j, \beta_j, B_j) \end{matrix} \right], \quad x > 0, \tag{39}$$

When this occurs, the Laplace transform of the compound sum S is transformed into

$$\mathcal{L}_S(s) = G_N(\mathcal{L}_X(s)), \tag{40}$$

Where is the function that generates N, and is the Laplace transformation of If N~Poisson(λ), for example, then

$$f_Y(y) = \int_0^{\infty} f_{Y|X}(y|x)\, g(x)\, dx, \tag{41}$$

and since $\mathcal{L}_X(s)$ often has an I-function representation, the inverse Laplace transform of produce an expression that still uses the I-function, either precisely or almost.

Likewise, in the case of mixed models, where

$$f_Y(y) = \int_0^{\infty} f_{Y|X}(y|x)\, g(x)\, dx, \tag{42}$$

If g(x) is represented by an I-distribution, the ensuing marginal distribution $f_Y(y)$ may similarly be articulated using the I-function, particularly when the conditional density $f_{Y|X}(y|x)$ has a manageable structure. This paradigm is extensively used in survival models including frailty and Bayesian models with hierarchical structures.

CONCLUSION

The Generalized I-function plays a pivotal role in advancing statistical modeling by offering a highly flexible and comprehensive framework for representing complex distributions. In statistics, the I-function enables the construction of new probability distributions (e.g., I-Weibull, I-Gamma) that better fit empirical data, supports robust Bayesian inference through the design of non-standard priors, and facilitates accurate moment analysis via Mellin transforms. Furthermore, it enhances modeling in risk theory through compound and mixture distributions, and aids in entropy-based statistical mechanics by representing maximum entropy distributions. Overall, the Generalized I-function provides statisticians with a powerful and adaptable toolset for analyzing complex data structures, improving model fit, and capturing intricate probabilistic behaviors.

The generalized I-function models complicated statistical distributions in a coherent and flexible manner. It may integrate many special functions and expand their application by adding scaling, skewness, and tail behavior factors due to its structural richness. These qualities make I-function-based distributions useful for assessing real-world data with anomalies that traditional models miss. Applications include dependability, survivability, Bayesian modeling, information theory, risk analysis, and thermodynamic entropy modeling. Construction of complicated statistical distributions is flexible using the generalized I-function. It generalizes classical functions like Meijer G and Fox H to handle scaling, skewness, and heavy tails more precisely. Distributions of form

$$f(x) = k \cdot x^{\rho-1} I_{p,q}^{m,n} \left[\alpha x^{\beta} \middle| \begin{matrix} (a_i, \alpha_i, A_i) \\ (b_j, \beta_j, B_j) \end{matrix} \right]$$

can handle real-world data from banking, reliability engineering, hydrology, and statistical physics.

To further our understanding of our work, we shall change various derivatives or special functions in order to conduct advanced research. In addition to this, we will use fractional differential equations to a wide range of scientific and engineering fields, including electromagnetic, fluid mechanics, and biological population models, amongst others.

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