Journal of Advances in Science and Technology
Vol. 22, Issue No. 2, April-2025, ISSN 2230-9659

DOI: https://doi.org/10.29070/18yyc224

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Study of generalised hypergeometric functions and fractional calculus with applications

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Abstract: A strong foundation for tackling complicated issues in engineering, physics, and applied sciences is offered by the study of generalised hypergeometric functions and fractional calculus, which is a major junction of classical analysis and current mathematical theory. The generalised hypergeometric functions are naturally generated in the solutions of diverse differential equations and have rich analytical features. They expand the classic hypergeometric functions over a bigger parameter space. For memory-dependent and hereditary phenomena that cannot be effectively represented by standard integer-order calculus, fractional calculus offers more sophisticated tools by dealing with integrals and derivatives of arbitrary (non-integer) order. An examination of the foundations and relationships between fractional calculus and generalised hypergeometric functions, as well as their operational methods, integral transforms, and special function representations, is the goal of this research. An analysis of fractional differential equations analytically solved by generalised hypergeometric functions is highlighted, illustrating their practical use in problems of viscoelasticity, anomalous diffusion, and signal processing, among others. By providing concrete examples, we go deeper into the applications and show how these mathematical tools have contributed to theoretical advancements as well as practical applications. The importance of fractional-order operators and special functions in modern scientific research and mathematical modelling is highlighted by this study, which ultimately adds to our knowledge of their synergy.

Keywords: Generalised, Hypergeometric Functions, Fractional, Applications

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INTRODUCTION

An intriguing extension of classical calculus, fractional calculus brings the idea of integrals and derivatives to orders that are not integers, namely fractions. This area of mathematical analysis has come a long way from its late 17th-century beginnings in a correspondence between Leibniz and L'Hôpital; since then, it has flourished and discovered many theoretical and practical uses in engineering and science. Fractional calculus permits operations of arbitrary real or complex order, in contrast to traditional calculus's operation with integrals and derivatives of integer order. This permits a more sophisticated and versatile description of memory, hereditary traits, and complex dynamics found in many systems, both natural and artificial. It has become an essential analytical tool in many different areas because to its mathematical complexity and adaptability, including control theory, bioengineering, signal processing, finance mathematics, and physics.
A group of special functions known as generalised hypergeometric functions provide solutions to many different kinds of linear differential equations; their theory has progressed in tandem with fractional calculus. The classical hypergeometric function incorporates several elementary and special functions such the exponential, logarithmic, trigonometric, and Bessel functions; the generalised hypergeometric function, abbreviated as pFq, is an extension of this. These functions are optimal for describing solutions to differential equations, including those in fractional calculus, because they are defined by power series with

coefficients containing ratios of gamma functions. The use of generalised hypergeometric functions to express or approximate solutions to fractional differential equations (FDEs) often results in a strong interaction between the two types of mathematical structures (W.K. Mohammed 2018).

The capacity of fractional calculus to more faithfully represent physical and biological processes than classical models is the driving force behind its development. Systems with non-local features, such as anomalous diffusion, viscoelastic behaviour, or long-range temporal or spatial dependencies, are generally intractable using integer-order models but may be solved with fractional models. Because fractional derivatives inherently include the history of a function's behaviour, they are well-suited for modelling systems with memory effects, in contrast to integer-order derivatives, which are local and memoryless. In fields like materials science, neurobiology, control systems, and even finance, where past data affects present dynamics, this quality is of paramount importance. The formulation of fractional calculus through several definitions, such as the Riemann-Liouville, Caputo, Grünwald-Letnikov, and Hadamard definitions, each with its own theoretical relevance and practical applicability, is one of the most fascinating elements of the field. Since it is compatible with beginning conditions described in terms of integer-order derivatives, the Caputo derivative is frequently preferred in practical issues among these. The shape of solutions using generalised hypergeometric functions is typically dictated by the choice of definition, which effects the analytical and numerical handling of problems. Integral transforms, Laplace and Fourier transforms, series solutions, and fractional differential equations all naturally produce these unique functions, demonstrating how useful they are as a link between theory and practice (D. F. Torres 2009).

HISTORICAL DEVELOPMENT OF CALCULUS

Among the greatest accomplishments in the annals of mathematics is the establishment of calculus. It sprang out of ancient civilisations' (Egypt, Greece, and India included) centuries-long obsession with understanding the fundamentals of motion, change, and area under curves. Both Sir Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716) worked independently but at the same time in the 17th century to formally conceptualise and rigorously develop calculus. Newton's method, which he called "fluxions," was inspired by physics issues, namely those involving gravity and movements. The notation scheme that contemporary calculus is based on was created by Leibniz, who also introduced the well-known symbols for differential (dx) and integral (∫).

Calculus relies on two main branches: differential and integral calculus. The idea of the derivative, which quantifies the rate of change of a variable at a given instant, is central to differential calculus. Here is the formal definition of the derivative of the function f(x) at the position x (D. F. Torres 2010):

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x)}$$

Two of calculus's most fundamental concepts are differential and integral calculus. As a measure of the rate of change of a quantity at a given moment, the derivative is central to the field of differential calculus. The derivative of a function f(x) at a given point x is classically defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where the Riemann sum, which is a way to approximate the area under a curve by adding the areas of narrow rectangles, is represented on the right-hand side.

A beautiful bridge between the two domains was laid down by the founders of calculus in the late 17th century with their Fundamental Theorem of Calculus. According to it, integration and differentiation are two sides of the same coin:

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x) and \int_a^b f'(x)dx = f(b) - f(a)$$

The ability to accurately represent natural occurrences was made possible by these findings, which caused a revolution in many fields, including mathematics, physics, astronomy, and engineering (Almeida, R. 2011).

Although calculus is officially born with the contributions of Newton and Leibniz, its foundations were laid much earlier. An early kind of integral calculus, the method of exhaustion was devised by Eudoxus (c. 408-355 BC) and subsequently employed by Archimedes (c. 287-212 BC). Utilising infinitesimally minute measurements, Archimedes approximated the area of a circle and the volume of a sphere (T. Mansour (2017). Likewise, mathematicians from India, such as Bhāskara II (1114–1185) and the Kerala School, like Madhava of Sangamagrama (c. 1350–1425), have made great strides in comprehending infinite series and differentiation, as well as in approximating sine and cosine functions using power series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

The subsequent creation of the Taylor and Maclaurin series in Europe was preceded by these series, which laid the groundwork for current analysis.

Even into the 18th and 19th centuries, calculus's formalisation of boundaries and rigour persisted. The ε-δ concept of a limit was proposed by mathematicians like Augustin-Louis Cauchy (1789–1857) and Karl Weierstrass (1815–1897), who eliminated the intuitive and occasionally nebulous usage of infinitesimals that had afflicted previous calculus. Their contributions guaranteed the consistent and reasonable application of calculus and prepared the way for serious analysis (S. Pilipovic´ 2008).

As calculus developed, new subfields appeared. The fascinating notion of fractional calculus is an expansion of integrals and derivatives from integer-order to non-integer (real or complex) orders. In a letter he sent in 1695, Leibniz asked what it may imply to have a derivative of order ½. This letter laid the groundwork for fractional calculus. Liouville, Riemann, and Caputo were among the mathematicians who later formalised this apparently counterintuitive notion.

The function f(t)'s Riemann-Liouville fractional integral of order $\alpha > 0$ is given by:

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

In this context, $\Gamma(\alpha)$ represents the Gamma function, which is an extension of the factorial function. For positive integers α, this operator simplifies to the usual n-fold integral. The Riemann-Liouville derivative, which is the fractional derivative in this case, is calculated as (Ahmad, S. 2013) (D. Baleanu 2016):

$$(D^{\alpha} f)(x) = \frac{d^n}{dx^n} (I^{n-\alpha})(x) \quad where\, n = \lceil \alpha \rceil$$

Since it permits starting conditions in terms of integer-order derivatives, the Caputo derivative—another popular definition—is more appropriate for physical applications:

$$c_{D^{\alpha}} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt \qquad with\, n = \lceil \alpha \rceil$$

These advancements highlight how calculus has progressed from a tool for tackling physical and geometrical issues to a rigorous and flexible framework that can explain memory and hereditary complicated dynamical systems.

EMERGENCE OF FRACTIONAL CALCULUS

Surprisingly, fractional calculus has a rich and lengthy history that dates back to before and even before many advancements in classical calculus. It is an extension of regular calculus to non-integer orders of differentiation and integration. While Newton's and Leibniz's 17th-century formalisation of calculus centred on integer-order integration and differentiation, the inquiry into the possibility of considering a derivative of arbitrary (non-integer) order arose nearly simultaneously with the introduction of classical definitions (S. Salahshour 2017).

In the 1830s, Joseph Liouville made the first mathematical attempt to define a non-integer order derivative by expanding the idea of repeated integration into fractional orders. Liouville is said to have given the first rigorous expression of a fractional integral, which was further developed by Riemann. In the beginning, people tried to find a way to make the Cauchy formula for repeated integration more generic. A function's n-fold integral over the interval [0, x] is given by for an integer n:

$$I^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt$$

Riemann and Liouville further expanded this equation to create a fractional integral of order $\alpha > 0$, as (P. J. Torvik 2000):

$$(I^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha - 1} f(t) dt$$

For positive integers n, Γ(n) = (n - 1)!, the Gamma function, which is a continuous extension of the factorial function, is denoted by Γ(α) below. This operator permits a smooth interpolation between integer orders, but it reduces to the classical integral when α is a positive integer. The basic idea behind the Riemann-Liouville fractional integral is to utilise the kernel (x - t)^{α - 1}, which represents the memory-like quality of fractional systems. This means that the value of the integral at x is dependent on the whole history of the function f(t) spanning the interval [0, x].

It is possible to obtain the Riemann-Liouville derivative of order α > 0 from the fractional integral (K. Nisar 2018):

$$(D^{\alpha} f)(x) = \frac{d^n}{dx^n} (I^{n-\alpha} f)(x), \;\; where\; n = \lceil \alpha \rceil$$

Despite being consistent theoretically, this concept had difficulties in physical applications. This was mainly due to the fact that it demanded that the function f(x) be differentiable up to order n, and the beginning conditions for differential equations were problematic when it came to physical values. In response to this, Michele Caputo proposed a new concept in the 1960s that worked better with engineering and physics initial value problems. This is the definition of the Caputo fractional derivative of order α > 0:

$$c_{D^{\alpha}} f(x) = \frac{1}{\Gamma(n - \alpha)} \int_0^x \frac{f^{(n)}(t)}{(x - t)^{\alpha - n + 1}} dt \qquad where\; n = \lceil \alpha \rceil$$

The main benefit of the Caputo derivative is that it helps to better align with empirical data and classical differential equation theory by allowing beginning conditions to be expressed in terms of integer-order derivatives. Because of this, Caputo's formulation has taken over as the go-to model in the applied sciences, particularly for systems with memory or genetic features, such as anomalous diffusion, viscoelasticity, and others (M. Fabrizio 2015).

In the late 19th and early 20th centuries, mathematicians like Abel, Heaviside, Grünwald, and Letnikov laid the groundwork for fractional calculus, which eventually became a valid mathematical topic. The tautochrone issue, which Abel studied, inevitably gave rise to integral equations using fractional operators. More specifically, his integral equation:

$$f(t) = \int_0^t \frac{\phi(\mathcal{T})}{t - \mathcal{T}^{1-\alpha}} d\mathcal{T}$$

immediately arrived to formulas that are a reflection of the present Riemann-Liouville integral. Simultaneously, the Grünwald-Letnikov method defined the fractional derivative utilising limits of difference quotients in a way that was more computationally friendly:

$$D^{\alpha}f(x)=\lim_{h\to 0}\frac{1}{h^{\alpha}}\sum_{k=0}^{\infty}(-1)^{k}\binom{\alpha}{k}f(x-kh)$$

Particularly useful in numerical simulations, this formulation bridges fractional calculus with discrete approximations (Caputo, M. 2016).

EVOLUTION OF SPECIAL FUNCTIONS AND HYPERGEOMETRIC FUNCTIONS

The theory of special functions, especially hypergeometric functions, is fundamental to both classical and contemporary mathematical analysis, offering crucial instruments for resolving intricate differential equations and simulating diverse physical events. Special functions historically originated not as abstract concepts but as answers to practical challenges in physics, astronomy, and engineering—numerous of which defied closed-form representation using fundamental functions. For centuries, mathematicians have systematically classified and analysed functions that frequently arise in mathematical physics, ultimately establishing a diverse category known as special functions, which encompasses the gamma function, beta function, Bessel functions, Legendre polynomials, Hermite polynomials, and, more generally, hypergeometric functions (I. O. Kıymaz 2016).

The historical origins of special functions can be dated to antiquity. The gamma function Γ(z) is a generalisation of the factorial function for complex numbers, initially examined in the 18th century by Leonhard Euler. Euler characterised the gamma function as an improper integral:

$$\Gamma(z)=\int_{0}^{\infty}t^{z-1}e^{-t}\,dt,\quad \Re(z)>0$$

that fulfils the recurrence connection $\Gamma(z+1)=z\Gamma(z)$, expanding the scope of factorials beyond integers. The beta function B(x, y) is defined as follows:

$$B(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}\,dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

These integrals established the groundwork for subsequent advancements in integral transformations and analytical continuation. During the 18th and 19th centuries, the exploration of definite integrals and infinite series revealed that numerous physical systems and boundary value problems produced solutions that could not be expressed using elementary functions, but rather corresponded to novel, "special" functions (R. Rani 2018).

A crucial juncture in this developmental trajectory was the formulation of the hypergeometric function, first examined by Euler and subsequently formally defined by Carl Friedrich Gauss. The traditional Gauss hypergeometric function is defined by the series:

$$2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} . \frac{z^n}{n!}$$

where (q)_n represents the Pochhammer symbol or rising factorial: $(q)_n = q(q+1)(q+2) \ldots\ldots (q+n-1)$ for . $n \geq 1, and\ (q)_0 = 1$. This series converges absolutely for |z| < 1 and defines a function that adheres to the hypergeometric differential equation:

$$z(1-z)\frac{d^2y}{dz^2} + [c - (a+b+1)z]\frac{dy}{dz} - aby = 0$$

This equation emerges in several contexts, including quantum physics, potential theory, and spherical harmonics. Gauss's significant insight was that the behaviour of several ostensibly unrelated functions could be consolidated under the hypergeometric function. Many classical functions, including logarithms, trigonometric functions, and inverse trigonometric functions, can be derived as specific or limiting examples of the hypergeometric series.

Inspired by Gauss, mathematicians expanded the function to incorporate more parameters and broader convergence areas. This resulted in the formulation of the generalised hypergeometric function, represented by:

$$pF_q\left(a_1, \ldots, a_p; b_1, \ldots\ldots, b_q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots.(a_p)_n}{(b_1)_n \ldots.(b_q)_n} . \frac{z^n}{n!}$$

where a_i and b_j are complex parameters, and the series converges for all z when p ≤ q, and for |z| < 1 when p = q + 1. When p > q + 1, the series diverges for any non-zero z, however it may still establish a function using analytic continuation. These generalised functions encompass a wide array of special functions, including Bessel functions, Legendre functions, Laguerre polynomials, and others. The Bessel function of the first kind, which resolves Bessel's differential equation and typically appears in situations exhibiting cylindrical symmetry, may be expressed as:

$$J_v(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+v+1)} \left(\frac{z}{2}\right)^{\frac{2n}{v}} = \frac{(\frac{Z}{2})^v}{\Gamma(v+1)} . 0F_1\left(; v+1; -\frac{z^2}{4}\right)$$

The confluent hypergeometric function, also known as Kummer's function, arises as a limiting case of the Gauss hypergeometric function when one singularity is extended to infinity. It is characterised by (S.M. Zubair 1994):

$$1F_1(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} . \frac{z^n}{n!}$$

and fulfils the differential equation:

$$z\frac{d^2y}{dz^2}+(c-z)\frac{dy}{dz}-ay=0$$

This function is present in the solutions of Schrödinger's equation for the hydrogen atom and other quantum systems characterised by radial potentials. The significance of these functions is amplified by the examination of orthogonal polynomials, including Hermite, Chebyshev, and Jacobi polynomials, many of which are specific instances of hypergeometric or confluent hypergeometric functions.

The 20th century had a swift proliferation in the comprehension and utilisation of special functions, propelled by advancements in mathematical physics and complicated analysis. The advent of Meijer G-functions and Fox H-functions significantly generalised hypergeometric structures. These functions possess the notable characteristic of include almost all classical special functions as particular instances. The Meijer G-function is characterised by a Mellin–Barnes type contour integral and provides a cohesive framework for addressing a wide range of integral transformations and fractional differential equations. It is frequently expressed as (S. M. Zubair 1997):

$$G_{p,q}^{m,n}\left(z\middle|\begin{matrix}a_1,\ldots\ldots,a_p\\ b_1,\ldots\ldots,b_q\end{matrix}\right)=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}z^{-s}ds$$

This function is essential in fractional calculus, where differential equations frequently provide solutions that classical functions cannot represent. In the modelling of viscoelastic materials or anomalous diffusion, solutions to fractional differential equations frequently use Mittag-Leffler functions, which are specific instances of generalised hypergeometric series:

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}=1F_1(1;\alpha+1;z$$

These functions generalise the exponential function (which is obtained when α = 1) and are crucial in the analytical resolution of fractional-order systems.

CONCLUSION

Finally, the exploration of generalised hypergeometric functions in the context of fractional calculus unveils a complex mathematical terrain endowed with deep theoretical and practical consequences. For many special functions in mathematical physics, engineering, and applied analysis, generalised hypergeometric functions provide a strong unifying foundation. The flexibility with which they may represent answers to complicated differential and integral equations makes them fundamental in contemporary mathematical modelling. When it comes to modelling memory-dependent and hereditary processes in areas like bioengineering, signal processing, and viscoelasticity, fractional calculus provides a versatile and precise solution. It does this by expanding classical calculus to non-integer orders. They both provide strong analytical tools for solving issues that traditional techniques can't seem to grasp when they come together. Integrating fractional derivatives using generalised hypergeometric functions paves the door to closed-form solutions to fractional differential equations, expanding the analytical toolbox for both theoretical and practical fields. When conducted as a whole, these studies improve our grasp of mathematical theory and our capacity to deal with complicated dynamical processes in the actual world. Taken together, these subjects demonstrate the finesse and practicality of sophisticated mathematical constructions in dealing with both theoretically-oriented questions and real-world problems.

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