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The extended jacobi polynomials in two variables: A generalization

Asha Pandey [1] * , Dr. Neelam Pandey [2]

1. Research Scholar, Govt. Model Science College, APS University, Rewa, M.P. , India
ashupandey4545@gmail.com ,
2. Professor of Mathematics, Govt. Model Science College, Rewa, APS University, Rewa, M.P. , India

Abstract: This work presents an extension of the Jacobi polynomials to two variables and generates a number of generating functions. In addition, certain applications of Jacobi polynomials are examined, along with Bateman's and Brafman's generating functions, Rodrigues formula, and the relationship between Legendre and Jacobi polynomials. The main objective of this study is to construct and describe certain properties of an analogue of extended Jacobi polynomials that operates on two variables. The authors provide recurrence relations that use extended □Jacobi polynomials in two variables, along with a variety of differential equations for these polynomials.

Keywords: Polynomials, Jacobi polynomials, two-variable, generalizations, Jacobi matrix polynomial

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INTRODUCTION

The conventional Jacobi polynomials' generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(z)t^n = 2^{\alpha+\beta}\varrho^{-1}(1-t+\varrho)^{-\alpha}\,(1+t+\varrho)^{-\beta}\,,$$

,

Were

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\left(\frac{z-1}{2}\right)^m,$$

The Legendre, Chebyshev, and Gegen Bauer (ultraspherical) the extensions of polynomials that belong to the family of orthogonal polynomials. Furthermore, the traditional Jacobi orthogonal polynomials have been crucial in a wide range of mathematical, physics, and engineering applications.

In contrast, Polynomials with matrix parameters and special functions have extensive use in engineering, probability theory, physics, statistics, and mathematical analysis. The Jacobi matrix polynomial (JMP) is one specific unique matrix polynomial that has been used a lot in recent studies.

$\mathcal{J}_n(E,F,z,w)$ The structure for two- Polynomials of the Kornhauser matrix two-variable polynomials of Shivley's matrix Quadratic Laguerre matrices the two-part Generalized Hermite matrix polynomials, Zweivariable Gegen Bauer matrix polynomials, and Two-Variable Chebyshev matrix polynomials of Gen 2, as well as their properties, have been proposed.

For the numerical analysis of many computing techniques, including spectrum and spectral element approaches, systems of orthogonal polynomials are crucial tools and the references therein). The most often used class of polynomials on limited domains is the Jacobi class, which contains ultraspherical polynomials. Numerous branches of mathematics, including spectral methods in PDE, high-order ordinary differential equation solving techniques, PDE systems, approximation theory, and finite element approaches, involve Jacobi polynomials. Their primary contribution to spectral methods is the spherical functions they represent on compact rank-one symmetric spaces and their strong coupling with the Laplace operator on these spaces. Hence, it is essential that they detail the Laplacian's spectral projections and spectral measure. They find use, for instance, in data compression, signal processing in condensed matter physics, and high-energy physics, and tensor fields in geophysical and astronomical issues efficiently evaluated, among other applications in mathematical physics and practical research.

Within Askey's hypergeometric orthogonal polynomial scheme hierarchy, the Jacobi polynomials, represented by $P_n^{(\alpha,\beta)}(x)$ have one more degree of freedom than the ultraspherical (or Gegenbauer) polynomials; as a result, Furthermore, Jacobi's includes Legendre and Chebyshev polynomials for example. New radial and angular coordinates are introduced in our work along with an integral representation for the Jacobi polynomials of the Dirichlet-Mehler type, which results in an updated version of Euler's formula in these specific coordinates. It is upon Koornwinder's work that this new integral representation is based. By using this new form to express the fractional integral, we may improve the Askey scheme and get the Jacobi polynomials expressed in terms of Gegenbauer polynomials. Adding a further step to Askey's method is as simple as putting Hahn's polynomials in terms of Jacobi's.

Dirichlet-Mehler's innovative depiction of $P_n^{(\alpha,\beta)}(x)$ Through demonstrating that the fractional integral of a measurable function's Fourier cosine coefficients may be represented by the function's suitably renormalized Jacobi spectral coefficients, the known strict relationship between fractional calculus and Jacobi polynomials is once again made manifest. From this vantage point, it seems that the Fourier-Jacobi coefficient representation we've settled on is particularly well-suited to the spectrum analysis of FDEs. Additionally, by using appropriate Fourier cosine coefficients to represent the spectral coefficients, we may take use of the precision and efficiency of the Fast Fourier Transform, which has computational advantages, especially for high values of n.

LITERATURE REVIEW

Aktaş, Rabia & Erkus-Duman, Esra. (2015). An extended Jacobi polynomial representation in two variables and a selection of its properties are the main aims of this study. The authors provide recurrence relations that use extended Jacobi polynomials in two variables, along with a variety of differential equations for these polynomials. Different families of bilateral and bilinear generating functions are derived. Additionally, this research presents a few unique circumstances of the findings.

Genest, Vincent & Lemay, Jean-Michel & Vinet, Luc & Zhedanov, Alexei. (2015). We introduce and clarify the extension of the Big −1 Jacobi polynomials to two variables. These bivariate polynomials are the product of two univariate Big-1 Jacobi polynomials. In this method, we may determine their orthogonality.

Lewanowicz and Wo'zny's two-variable huge q-Jacobi polynomials are examined using the limiting approach to ascertain their bispectral properties using equations of the Pearson type is another way to get the weight function.

Xu, Yuan. (2014). A generating function for Hahn polynomials of many variables may be found in the Jacobi polynomials on the simplex. By following this path, we may learn a few things about these two classes of orthogonal polynomials. The Hahn polynomials are shown to occur as connection coefficients among a number of orthogonal polynomial families on the simplex. We provide closed-form equations for the replicating kernels of the Krawtchouk and Hahn polynomials. As an application, The Krawtchouk and Hahn polynomials' Poisson kernels are shown to be nonnegative.

Ali, Asad. (2021). The generalised Laguerre polynomial for two variables is introduced and studied in this work. We demonstrate that the generalised hypergeometric function characterises these polynomials. Some recurrence relations, generating functions, and an explicit representation are shown. Additionally, these polynomials show up as differential equation solutions.

JACOBI POLYNOMIALS OF TWO VARIABLES

Polynomials of Jacobi for two variables $P_n^{(\alpha,\beta)}(x)$, may be defined as:

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{yt}+\rho\right)^{-\beta}\left(1-\sqrt{yt}+\rho\right)^{-\alpha} \ \ldots\ldots (1)$$

Were

$$\rho = (1-2xt+yt^2)^{\frac{1}{2}}$$

Derivation: P. 270(2) defines Jacobi polynomials.

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,t^n = F_4\left(1+\beta,1+\alpha;1+\alpha,1+\beta;\tfrac{1}{2}t(x+1),\tfrac{1}{2}t(x+1)\right) (2)$$

Additionally, we remember the outcome:

If neither a nor b is a negative integer or 0,

$$F_4\left(a,b;b,a;\frac{-u}{(1-u)(1-v)},\frac{-v}{(1-u)(1-v)}\right) = (1-uv)^{-1}(1-u)^a(1-v)^b \ldots\ldots.. (.3)$$

where a =1+β, b = 1+α

$$\frac{-u}{(1-u)(1-v)} = \frac{t(x-1)}{2}, \frac{-v}{(1-u)(1-v)} = \frac{t(x+1)}{2} \ \ldots\ldots (.4)$$

Let's look at the Jacobi polynomial extension for two variables.

$$\rho = (1 - 2xt + yt^2)^{\frac{1}{2},}$$ and

$$U=1-\frac{2}{1+\sqrt{yt}+\rho}, \quad v=1-\frac{2}{1-\sqrt{yt}+\rho} \quad \ldots\ldots. (5)$$

Now from eq. (5)

$$\frac{-u}{(1-u)(1-v)} = \frac{1}{1-v}\left(1 - \frac{1}{1-u}\right)$$

$$= \frac{1-\sqrt{yt}+\rho}{2}\left(1 - \frac{1+\sqrt{yt}+\rho}{2}\right)$$

$$= \frac{(1-\sqrt{yt}+\rho)(1-\sqrt{yt}-\rho)}{4}$$

$$= \frac{(1-\sqrt{yt})^2-\rho^2}{4}$$

$$= \frac{t(x-\sqrt{y})}{2} \quad \ldots\ldots.. (6)$$

Similarly

$$\frac{-v}{(1-u)(1-v)} = \frac{t(x+\sqrt{y})}{2} \quad \ldots.. (7)$$

Further

$$\frac{1}{1-u} = \frac{1}{2}\left(1+\sqrt{yt}+\rho\right) \; and \; \frac{1}{1-v} = \frac{1}{2}\left(1-\sqrt{yt}+\rho\right)$$

from which we can have

$$\rho = \frac{1}{1-u} + \frac{1}{1-v} - 1$$

$$= \frac{1-uv}{(1-u)(1-v)}$$

and hence

$$(1-uv)^{-1}(1-u)^a(1-v)^b = \rho^{-1}(1-u)^{a-1}(1-v)^{b-1}$$

Thus from eq. (2.2) and eq. (2.3) we obtain

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = \rho^{-1}\left(\frac{2}{1+\sqrt{yt}+\rho}\right)^{\beta}\left(\frac{2}{1-\sqrt{yt}+\rho}\right)^{\alpha}$$

Or

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)\,t^n = 2^{\alpha+\beta}\rho^{-1}\left(1+\sqrt{yt}+\rho\right)^{-\beta}\left(1-\sqrt{yt}+\rho\right)^{-\alpha}.$$

Were

$$\rho = (1-2xt+yt^2)^{\frac{1}{2}}$$

This concludes the proof.

Additionally, using equations (2), (6), and (7), we get

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)t^n = F_4\left(1+\beta, 1+\alpha; 1+\alpha, 1+\beta; \tfrac{t}{2}\left(x-\sqrt{y}\right), \tfrac{t}{2}\left(x+\sqrt{y}\right)\right) \quad \ldots\ldots (8)$$

Equation (8) may be expressed as

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x,y)\,t^n = \sum_{n,k=0}^{\infty} \frac{(1+a)_{n+k}(1+\beta)_{n+k}\frac{1}{2}\left(x-\sqrt{y}\right)^k \frac{1}{2}\left(x+\sqrt{y}\right)^n t^n}{k!\,n!\,(1+a)_k(1+\beta)_n}$$

which might be made simpler to get the methods that generate $P)_n^{(\alpha,\beta)}(x,y)$ as:

$$P_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \frac{(1+\alpha)_n(1+\beta)_n}{k!(n-k)!(1+\alpha)_k(1+\beta)_{n-k}}\left(\frac{x-\sqrt{y}}{2}\right)^k\left(\frac{x+\sqrt{y}}{2}\right)^{n-k} \quad \ldots\ldots (9)$$

$$P_n^{(\alpha,\beta)}(x,y) = \frac{(1+a)_n}{n!}\left(\frac{x+\sqrt{y}}{2}\right)^n 2F_1\left[\begin{matrix}-n, -\beta-n; \frac{x-\sqrt{y}}{x+\sqrt{y}} \\ 1+\alpha;\end{matrix}\right] \quad \ldots\ldots (10)$$

And

$$P_n^{(\alpha,\beta)}(x,y) = \frac{(1+\alpha)_n}{n!} 2F_1\left[\begin{matrix}-n, 1-\alpha+\beta+n; \frac{\sqrt{y}-x}{2} \\ 1+\alpha;\end{matrix}\right]. \quad \ldots\ldots (11)$$

For α = β = 0, (10) reduce to two-variable Legendre polynomials

$$P_n(x,y) = \left(\frac{x+\sqrt{y}}{2}\right)^n 2F_1\left[\begin{matrix}-n, -n; \frac{x-\sqrt{y}}{x+\sqrt{y}} \\ 1;\end{matrix}\right] . \quad \ldots\ldots (12)$$

Additionally, equation (11) provides a finite series form for $P_n^{\alpha,\beta}(x,y)$ as

$$P_n^{(\alpha,\beta)}(x,y) = \sum_{k=0}^{n} \frac{(1+a)_n(1+a+\beta)_{n+k}}{k!(n-k)!(1+a)_k(1+a+\beta)_n}\left(\frac{x-\sqrt{y}}{2}\right)^k \quad \ldots\ldots (13)$$

We examine the series in order to get a generating function for the Jacobi polynomials from eq. (13)

$$\psi = \sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n}$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(1+\alpha+\beta)_{n+k}\left(\frac{x}{2}-\frac{\sqrt{y}}{2}\right)^k t^n}{k!\,(n+k)!\,(1+\alpha)_n}$$

$$= \sum_{n,k=0}^{\infty} \frac{(1+\alpha+\beta)_{n+2k}\left(x-\sqrt{y}\right)^k t^{n+k}}{k!\,n!\,(1+\alpha)_k 2^k}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(1+a+\beta+2k)_n t^n}{n!} \frac{(1+\alpha+\beta)_{2k}\left(x-\sqrt{y}\right)^k t^k}{k!\,(1+\alpha)_k 2^k}$$

$$= \sum_{k=0}^{\infty} \frac{(1+\alpha+\beta)_{2k}(x-\sqrt{y})^k t^k}{k!2^k(1+\alpha)_k(1-t)^{1+\alpha+\beta+2k}}$$

We all know that:

$$(\alpha)_{2k} = 2^{2k}\left(\frac{1}{2}\alpha\right)_k\left(\frac{\alpha}{2}+\frac{1}{2}\right)_k$$

Thus, we obtain

$$\sum_{n=0}^{\infty} \frac{(\alpha+\beta+1)_n P_n^{(\alpha,\beta)}(x,y)t^n}{(1+\alpha)_n} = (1-t)^{-1-\alpha-\beta} 2F_1\left[\frac{1}{2}(1+\alpha+\beta), \frac{1}{2}(2+\alpha+\beta); \frac{2t(x-\sqrt{y})}{(1-t)^2}\right] \quad \ldots\ldots (14)$$

This generates the Jacobi polynomials in another way.

A TWO-VARIABLE EJP ANALOGUE AND PROPERTIES

In this paper, we develop a two-variable equivalent of the EJPs by using the equality and the two-variable Jacobi polynomials:

$$F_{n,k}^{y}(x,y;a,b,c) = \left(c(a-b)\right)^{-k} F_{n-k}^{\left(y+k+\frac{1}{2},y+k+\frac{1}{2}\right)}(x;a,b,c)$$

$$\times \left((x-a)(b-x)\right)^{k/2}$$

$$\times F_{k}^{(y,y)}\left(\frac{a+b}{2} - \frac{(a-b)^2 y}{\sqrt[4]{(x-a)(b-x)}};a,b,c\right) \qquad (15)$$

with degree n for $n\ k \geq \geq 0$.

Theorem 1. The polynomials $F_{n,k}^{y}(x,y;a,b,c)$ fulfil the differential equation that follows.

$$(x-a)(x-b)v_{xx} + y(2x-a-b)v_{xy} + (y^2-1)v_{yy}$$

$$+(2y+3)\left(\frac{2x-a-b}{2}v_x + yv_y\right) = n(n+2y+2)v.$$

Theorem 2. For the polynomials

$F_{n,k}^{y}(x,y;a.b,c)$ we have

(A)

$$F_{n,k}^{y}(x,y;a,b,c)$$

$$= \sum_{m=0}^{k}\sum_{l=0}^{n-k} A_{m,l}^{n,k}(a,b,c;y)$$

$$\times (x-a)^{l+\frac{k-m}{2}}(b-x)^{\frac{k-m}{2}}$$

$$\times \left(2\sqrt{(x-a)(b-x)} + (a-b)y\right)^{m}$$

Where

$$A_{m,l}^{n,k}(a,b,c;y)$$

$$= \frac{(-1)^m c^{n-k}(a-b)^{n-k-l}(1+y)_k \left(y+k+\frac{3}{2}\right)_{n-k}}{2^{2m} m!\,(k-m)!\,l!\,(n-k-l)!\,(1+y)_m \left(y+k+\frac{3}{2}\right)_l}$$

$$\times \frac{(2y+2k+2)_{n-k+l}(1+2y)_{m+k}}{(1+2y)_k(2y+2k+2)_{n-k}};$$

(B)

$$F_{n,k}^{\gamma}(x,y;a,b,c)$$

$$= \{c(a-b)\}^{n-k} \binom{\gamma+n+\frac{1}{2}}{n-k} \binom{\gamma+k}{k} \left((x-a)(b-x)\right)^{k/2}$$

$$\times\, 2F1\left(k-n, 2\gamma+k+n+2; \gamma+k+\frac{3}{2}; \frac{x-a}{b-a}\right)$$

$$\times\, 2F1\left(-k, 2\gamma+k+1; \gamma+1; \frac{1}{2} - \frac{(b-a)y}{\sqrt[4]{(x-a)(b-x)}}\right).$$

Theorem 3. The EJPs provided by (15) have a two-variable analogue that is orthogonal to the weight function

$$\omega(x,y;a,b,\gamma) = \left(1 - \left(\frac{2x-a-b}{b-a}\right)^2 - y^2\right)^{\gamma}$$

over the domain

$$\Omega: \left\{(x,y): \left(\frac{2x-a-b}{b-a}\right)^2 + y^2 \le 1\right\}.$$

Proof. By (15)

$$\iint_{\Omega} F_{n,k}^{\gamma}(x,y;a,b,c) F_{m,l}^{\gamma}(x,y;a,b,c)\,\omega(x,y;a,b,\gamma)\,dxdy$$

$$= \{c(a-b)\}^{-k-1} \iint_{\Omega} \Big\{ F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)$$

$$\times F_k^{(\gamma,\gamma)}\left(\frac{a+b}{2} - \frac{(a-b)^2}{\sqrt[4]{(x-a)(b-x)}}; a,b,c\right)\left((x-a)(b-x)\right)^{\frac{k+l}{2}}$$

$$\times F_{m-l}^{\left(\gamma+l+\frac{1}{2},\gamma+l+\frac{1}{2}\right)}(x;a,b,c) F_l^{(\gamma,\gamma)}\left(\frac{a+b}{2}\right.$$

$$\left. - \frac{(a-b)^2 y}{\sqrt[4]{(x-a)(b-x)}}; a,b,c\right)$$

$$\times \left(1 - \left(\frac{2x-a-b}{b-a}\right)^2 - y^2\right)^{\gamma} \Big\} dxdy$$

$$= \frac{4^{2\gamma+1}}{(b-a)^{4\gamma+2}} \{c(a-b)\}^{-k-l} \int_a^b F_{n-k}^{\left(\gamma+k+\frac{1}{2},\gamma+k+\frac{1}{2}\right)}(x;a,b,c)$$

$$\times F_{m-l}^{\left(\gamma+l+\frac{1}{2},\gamma+l+\frac{1}{2}\right)}(x;a,b,c)\left((x-a)(b-x)\right)^{\frac{k+l+1}{2}+\gamma} dx$$

$$\times \int_a^b F_k^{(\gamma,\gamma)}(u;a,b,c) F_l^{(\gamma,\gamma)}\left(u;a,b,c\left((u-a)(b-u)\right)^{\gamma}\right) du = 0$$

For$(n,k) \neq (m,l)$ Hence Proved.

Theorem 4. The polynomials' quadratic norm $F_{n,k}^{\gamma}(x,y;a,b,c)$ is determined as

$$\left\| F_{n,k}^{\gamma}(x,y;a,b,c) \right\|^2$$

$$= \iint_{\Omega} \left[F_{n,k}^{\gamma}(x,y;a,b,c)\right]^2 w(x,y;a,b,\gamma) dxdy$$

$$= \frac{2^{4\gamma+1}(b-a)^{2n+1} c^{2(n-k)} \Gamma^2\left(\gamma+n+\frac{3}{2}\right)}{(n-k)!k!(\gamma+n+1)(2\gamma+2k+1)}$$

$$\times \frac{\Gamma^2(\gamma+k+1)}{\Gamma(2\gamma+n+k+2)\Gamma(2\gamma+k+1)}$$

Proof. It can easily be proved by (15)

CONCLUSION

After classical Jacobi polynomials were extended to two- and three-variable equivalents, the theory of multivariate orthogonal polynomials made great strides. While accommodating the intricacies of higher-dimensional domains like triangles and simplices, these generalizations maintain important structural characteristics like orthogonality, recurrence relations, and generating functions.

From numerical analysis to the spectral solution of partial differential equations, two-variable Jacobi polynomials constructed across triangular domains have proven useful. In a similar vein, three-variable analogues built on the 3-simplex provide reliable instruments for developing spectral approaches in three-dimensional environments and approximating functions.

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