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REVIEW ARTICLE

A ROLE OF CONNECTED SETS IN MATHEMATICS

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A Role of Connected Sets in Mathematics

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INTRODUCTION

A connected set is a set that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Let X be a topological space. A connected set in X is a set $A \subseteq X$ which cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set A. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other. The space X is a connected topological space if it is a connected subset of itself.

The real numbers are a connected set, as are any open or closed interval of real numbers. The (real or complex) plane is connected, as is any open or closed disc or any annulus in the plane. The topologist's sine curve is a connected subset of the plane. An example of a subset of the plane that is not connected is given by

$$B = \{z \in \mathbb{C} : |z| < 1 \text{ or } |z - 2| < 1\}.$$

Geometrically, the set B is the union of two open disks of radius one whose boundaries are tangent at the number 1.

An open set **S** is called **disconnected** if there are two open, non-empty sets **U** and **V** such that:

1. **U** $\cap$ **V** = **0**
2. **U** $\cup$ **V** = **S**

A set **S** (not necessarily open) is called **disconnected** if there are two open sets **U** and **V** such that

1. (**U** $\cap$ **S**) # **0** and (**V** $\cap$ **S**) # **0**
2. (**U** $\cap$ **S**) $\cap$ (**V** $\cap$ **S**) = **0**
3. (**U** $\cap$ **S**) $\cup$ (**V** $\cap$ **S**) = **S**

If **S** is not disconnected it is called **connected**.

Note that the definition of disconnected set is easier for an open set **S**. In principle, however, the idea is the same: If a set **S** can be separated into two open, disjoint sets in such a way that neither set is empty and both sets combined give the original set **S**, then **S** is called disconnected.

To show that a set is disconnected is generally easier than showing connectedness: if you can find a point that is not in the set **S**, then that point can often be used to 'disconnect' your set into two new open sets with the above properties.

Hence, as with open and closed sets, one of these two groups of sets are easy:

- open sets in **R** are the union of disjoint open intervals
- connected sets in **R** are intervals

The other group is the complicated one:

- closed sets are more difficult than open sets (e.g. Cantor set)
- disconnected sets are more difficult than connected ones (e.g. Cantor set)

In fact, a set can be disconnected at every point.

A set S is called totally disconnected if for each distinct x, y S there exist disjoint open set U and V such that x U, y V, and (U S) (V S) = S.

Intuitively, totally disconnected means that a set can be be broken up into two pieces at each of its points, and the breakpoint is always 'in between' the original set.

Connected and Disconnected Sets

Definition. A subset A of a metric space (X, ρ) is said to be **disconnected** if there are two disjoint open sets U and V such that

(i) $U \cup V \supseteq A$, and

(ii) $U \cap A$ and $V \cap A$ are both non-empty.

Continuous Functions and Connected Sets

Theorem Let (X, ρ) and (Y, σ) be metric spaces and suppose that

$$f : X \longrightarrow Y$$

is a continuous function. If $A \subseteq X$ is connected, then $f(A)$ is connected in Y.

Proof.

We can combine our particularly good understanding of connected subsets in the real line with Theorem. On the face of it, all this says is that the continuous image of an interval is also an interval-cut, but perhaps not very useful.

However, if we combine this with the defining property of an interval, we obtain an extremely useful theorem indeed: the Intermediate Value Theorem.

We start by proving the theorem, and the remainder of the chapter will be taken up with two applications of this theorem: a quick construction of the logarithmic functions, and a method for calculating roots of equations.

Theorem **(Intermediate Value Theorem)** Let f be a continuous real-valued function defined on $[a, b]$ and suppose $f(a) < f(b)$. Then, given any value of y_0 with

$$f(a) < y_0 < f(b)$$

then we can find an $x_0 \in (a, b)$ such that

$$y_0 = f(x_0)$$

Proof. If $x \neq y$ then one or the other of $x < y$ or $y < x$ must hold. Supposing without loss of generality that $x < y$, then by Factoid 7 from the development of the exponential functions, $a^x < a^y$. Thus $f(x) \neq f(y)$ and so f is one-to-one.

We know that f maps into $(0, +\infty)$ and so we need to show that it maps onto. Let $y_0 > 0$ and find an x_0 such that $f(x_0) = y_0$.

Since $a > 1$, $\lim_{n \to +\infty} a^n = +\infty$, and so there is an $n \in \mathbb{N}$ such that $a^n > y_0$. Since, also $a^{-n} \to 0$, there is an $m \in \mathbb{N}$ such that $a^{-m} < y_0$. Since

$$a^{-m} < y_0 < a^n$$

by the Intermediate Value Theorem, there is an $x_0 \in (-m, n)$ such that $f(x_0) = a^{x_0} = y_0$.

THE METHOD OF BISECTION

The method of bisection is a very simple and effective means of estimating the roots of equations. Unlike other methods for finding roots, which may be faster, this is the *only* method which is guaranteed to work for all continuous functions.

The theoretical underpinning of the method of bisection is the Intermediate Value Theorem. This tells us that if we know that a function is positive at one point, and negative at another, then a root of the function (a point at which it is zero) must lie between these two points.

As an example of how to use this algorithm, we shall approximate the square root of 2 to within 0.1.

Let $f(x) = x^2 - 2$.

1. Note that $f(1) = 1 - 2 = -1 < 1$ and $f(2) = 4 - 2 = 2 > 0$, so, by the Intermediate Value Theorem, there is an $x_0 \in (1, 2)$ such that $f(x_0) = 0$.
2. Consider the midpoint of $(1, 2)$, which is $3/2$.
3. $f(3/2) = (9/4) - 2 > 0$ and we knew already that $f(1) < 0$, so, by the Intermediate Value Theorem, there is an $x_0 \in (1, 3/2)$ such that $f(x_0) = 0$.
4. Consider the midpoint of $(1, 3/2)$, which is $5/4$.
5. $f(5/4) = (25/16) - 2 < 0$ and we knew already that $f(3/2) > 0$, so, by the Intermediate Value Theorem, there is an $x_0 \in (5/4, 3/2)$ such that $f(x_0) = 0$.
6. Consider the midpoint of $(5/4, 3/2)$, which is $11/8$.
7. $f(11/8) = (121/64) - 2 < 0$ and we knew already that $f(3/2) > 0$, so, by the Intermediate Value Theorem, there is an $x_0 \in (11/8, 3/2)$ such that $f(x_0) = 0$.
8. Consider the midpoint of $(11/8, 3/2)$, which is $23/16$. Because $x_0 \in (11/8, 3/2) = (22/16, 24/16)$, this shows that $|x_0 - 23/16| < 1/16$.

Thus, we can say that $23/16$ is an approximation to $\sqrt{2}$ which is accurate to within $1/16$.

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