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Numerical solutions for nonlinear equations: Development and analysis of new iterative methods

Rohit Kumar Pandey[1]*, Dr. Jaya Kushwah[2]

1 Research Scholar, Dept. of Mathematics, Sardar Patel University Balaghat M.P.

rkpandeyjii@gmail.com

2 Associate Professor, Sardar Patel University, Balaghat, M.P.

Abstract

In scientific and technical computations, the numerical solution of nonlinear equations is essential, as analytical solutions are not always applicable or accessible. New iterative approaches to efficiently and accurately solve nonlinear equations of the type f(x)=0, f(x)=0 are the subject of this work, which also rigorously analyses existing methods. While Secant algorithms and Newton-Raphson techniques are frequently employed, they have some drawbacks when it comes to convergence speed, reliance on initial estimations, and sensitivity to the function's nature. In response to these difficulties, the study presents hybrid and modified iterative systems that, under relaxed settings, show better convergence characteristics, namely stability and speed. Without substantially raising computing cost, the suggested approaches are built utilising multipoint evaluations or higher-order derivatives. In order to determine the convergence order, error boundaries, and stability requirements, a thorough convergence study is conducted. By conducting thorough numerical tests on benchmark nonlinear equations and comparing the outcomes with preexisting classical procedures, we confirm the efficacy and resilience of the novel methods. Suitable for tackling complicated nonlinear problems encountered in real-world applications, the suggested iterative approaches offer a considerable increase in accuracy and efficiency, as demonstrated by the findings.

Keywords: Numerical Solutions, Nonlinear Equations, Development, Iterative Methods

1. INTRODUCTION

In many fields of science, engineering, and industry, nonlinear equations crop up, and finding precise analytical solutions can be a real challenge. Because of this, numerical approaches to solving certain types of problems are now standard techniques in computational mathematics. The Newton-Raphson, Secant, and Bisection methods are some of the most well-known and widely-used traditional iterative approaches to solving nonlinear equations (Adomian, G. 2019). Nevertheless, these traditional approaches frequently encounter issues including sluggish convergence, vulnerability to first assumptions, and ineffectiveness when dealing with many or unconditioned roots. In light of these difficulties, research into creating and evaluating novel iterative methodologies has recently emerged as a hot topic. By lowering computing cost, improving stability, and raising the order of convergence, these state-of-the-art methods seek to outdo previous approaches (Hernández, M. A. 2004). To get around the problems with classical methods, researchers have come up with new systems that are higher-order and multi-step, and that incorporate memory and derivative-free strategies. Performance testing against benchmark nonlinear functions, stability investigations, and thorough convergence analyses are important parts of this progress (Chun, C. 2005). These evaluations guarantee that the suggested approaches work well in practice and have solid theoretical foundations. Modern symbolic computation and computing power have also made it easier to build and execute sophisticated iterative algorithms. New iterative approaches are always being developed, which shows how numerical analysis is always changing and how important it is for solving difficult nonlinear problems efficiently in many fields (Loghmani, G. M. 2014).

2. METHODOLOGY

Approaches to solving the research problem in a methodical manner are referred to as research methodologies. It will be regarded as a scientific discipline that focusses on the study of research methodologies. The paper describes the rationale, procedures, and outcomes of the present investigation. Since well-conducted research removes doubt and explains the findings, it may help in better planning the study's targets and objectives. One of the most essential and difficult challenges in logic and design applications is finding agreements, nonlinear condition frameworks, and unconstrained improvement issues. To resolve these cases' inaccurate configurations, mathematical approaches based on cycle systems are necessary. It is likely that Newton's plan is the most well-known iterative way to resolving these issues. Updates to Newton's approach have been proposed in writing, and each of them is as good as, if not better than, the original.

2.1 Research Type

It is the analysis kind that decides the value of the study's data. The analysis will mostly focus on quantitative measurements, but the qualitative components will also be present in the continuing study to account for the data's features. Using the victims and the results described by the current proposal's qualitative investigation, the researcher will try to merge the conventional approaches and get rid of their variations, like Newton's plan.

2.2 Sample Design

Sampling is occasionally the sole realistic choice in certain scientific circumstances due to the impossibility of studying the whole universe. In this set of investigations, we shall not alter our nature. The protocols outlined in "Development and Analysis of Some New Iterative Methods for Numerical Solutions of Nonlinear Equations" will be followed in order to choose research samples.

3. RESULTS

3.1 convergence analysis

In next research, our fifth-order IM will accomplish semilocal as well as local convergence. For a certain operator F, the method being discussed may be expressed as follows: Ω ⊆ B1 → B2.

$$\begin{cases} y^{(k)} &= x^{(k)} - F'(x^{(k)})^{-1}F(x^{(k)}), \\ x^{(k+1)} &= y^{(k)} - F'(y^{(k)})^{-1}F(y^{(k)}) - Q(x^{(k)},y^{(k)})F'(y^{(k)})^{-1}F(y^{(k)}), \end{cases} \tag{3.1.1}$$

Where $Q(x^{(k)},y^{(k)}) = Q^{(k)}$ is a scalar function that is defined as $Q(x^{(k)},y^{(k)}) = \frac{F(y^{(k)})^T F(y^{(k)})}{F(x^{(k)})^T F(x^{(k)})}$ and the superscript 'T' represents the transpose of operator F. In particular for B1 = B2 = $\mathbb{R}^n$, $Q(x^{(k)},y^{(k)}) = \frac{\|F(y^{(k)})\|^2}{\|F(x^{(k)})\|^2}$.

Following the approach discussed earlier, 3.1.1 zeroes in on the local analysis of procedure 3.1.1. In Section 3.1.2, the semilocal analysis is set up.

Local convergence analysis

For the local convergence analysis of the technique to be set up, some real parameters and functions need to be added. Since S is the set [0,∞], let's pretend the following:

(i) There exists a function, φ0 : S → S, meaning that is both continuous and non-decreasing; so, the equation

$$\phi_0(t) - 1 = 0,$$

Has the smallest solution s0 ∈ S\{0}. Set S0 = [0, s0).

(ii) There exists a function, φ : S0 → S, If the function is continuous and does not decrease, and the equation,

$$\chi_1(t) - 1 = 0,$$

Has the smallest solution ρ1 ∈ S0\{0}, where χ1 : S0 → R is defined as

$$\chi_1(t) = \frac{\int_0^1 \phi((1-\theta)t)d\theta}{1-\phi_0(t)}. \qquad (3.1.2)$$

(iii) There exists a smallest positive solution s1: ∈ S0\{0} of the equation,

$$\phi_0(t\chi_1(t)) - 1 = 0.$$

Set s = min{s0,s1} and S1 = [0,s).

(iv) There exists a continuous and non-decreasing function ψ : S1 × S1 → S such that the equation,

$$\chi_2(t) - 1 = 0,$$

Has the smallest root ρ2 ∈ S1\{0}, where χ2 : S1 → R is defined as

$$\chi_2(t) = \frac{\int_0^1 \phi((1-\theta)t\chi_1(t))d\theta + \psi(t,t\chi_1(t))\left(1+\int_0^1 \phi_0(\theta t\chi_1(t))d\theta\right)}{1-\phi_0(t\chi_1(t))}\chi_1(t). \qquad (3.1.3)$$

Define ρ ∗ ∈ S1\{0} by

$$\rho^* = \min\{\rho_1, \rho_2\}. \qquad (3.1.4)$$

Our objective is to demonstrate that the convergence radius for method is ρ ∗. Assuming that S* = [0,ρ*), we may deduce, for any t ∈ S*,

$$0 \le \phi_0(t) < 1, \qquad (3.1.5)$$

$$0 \le \chi_1(t) < 1, \qquad (3.1.6)$$

$$0 \le \chi_2(t) < 1. \qquad (3.1.7)$$

Take x to be a point in Ω and consider B[x, r] as the set that contains all points inside an open ball B(x, r) with a radius of r. It is necessary for the operator F to satisfy the following requirements (C1)-(C5) before building the major result on the local analysis may proceed:

(C_1): Equation defined by (4.1.1), i.e. $F(x) = \mathbf{0}$, has a particular solution $x^* \in \Omega$ such that the inverse operator $F'(x^*)^{-1} \in L(B_2, B_1)$.

(C_2): For each $x \in \Omega$,

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq \phi_0(\|x - x^*\|).$$

Set $\Omega 1 = B(x*, s0) \cap \Omega$.

(C3): For each x, y ∈ Ω1,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \phi(\|x - y\|).$$

Set $\Omega 2 = B(x*, s) \cap \Omega$.

(C4): For each x, y ∈ Ω2,

$$\|Q(x,y)\| \leq \psi(\|x - x^*\|, \|y - x^*\|).$$

(C5): B[x∗, ρ∗] ⊂ Ω.

Using the criteria (C1)-(C5), the next section develops the local convergence analysis for the technique under consideration.

Proposition: Assuming that the initial estimate x(0) ∈ B(x*, ρ*)\{x*} is selected under conditions (C1)-(C5), the iterative procedure described by equation remains valid for all k = 0, 1, 2,...

$$x^{(k)} \in B(x^*, \rho^*), \tag{3.1.8}$$

$$\|y^{(k)} - x^*\| \leq \chi_1(\|x^{(k)} - x^*\|)\, \|x^{(k)} - x^*\| < \|x^{(k)} - x^*\| < \rho^*, \tag{3.1.9}$$

$$\|x^{(k+1)} - x^*\| \leq \chi_2(\|x^{(k)} - x^*\|)\ \|x^{(k)} - x^*\| < \|x^{(k)} - x^*\|, \tag{3.1.10}$$

Where the functions χ1 and χ2 are defined in Eqs. (4.2.2) and (4.2.3), respectively, and ρ ∗ is defined in Eq. (4.2.4). Furthermore $\lim_{k\to\infty} x^{(k)} = x^*.$

Proof: Through the use of mathematical induction on 'k,' the assertions (3.1.8)-3.1.10) will be demonstrated. For k = 0, Eq. (3.1.8) is valid according to the supposition that x (0) belongs to B(x*,ρ *)\{x* }. The point u ∈ B(x ∟, ρ *)\{x* } can be picked at random. By extension, we may deduce from the condition (C2) and Eqs. (3.1.4), (3.1.5) that

$$\|F'(x^*)^{-1}(F'(u) - F'(x^*))\| \leq \phi_0(\|u - x^*\|) \leq \phi_0(\rho^*) < 1. \tag{3.1.11}$$

An invertible linear operator Banach lemma, together with Eq. (3.1.11), imply that $F'(u)^{-1} \in \mathscr{L}(\mathscr{B}_2, \mathscr{B}_1)$ and

$$||F'(u)^{-1}F'(x^*)|| \leq \frac{1}{1 - \phi_0(||u - x^*||)}. \tag{3.1.12}$$

As a consequence of (3.1.12), in particular for u = x(0), the iterate y(0) exists which is well-defined by the first step of (3.1.1) for k = 0. Then, it follows that

$$\begin{aligned} y^{(0)} - x^* &= x^{(0)} - x^* - F'(x^{(0)})^{-1}F(x^{(0)}) \\ &= F'(x^{(0)})^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}\left[F'(x^{(0)}) - F'(x^* + \theta(x^{(0)} - x^*))\right](x^{(0)} - x^*)d\theta. \end{aligned} \tag{3.1.13}$$

Using Eqs. (3.1.4), (3.1.6), (3.1.12) (for $u = x^{(0)}$) and condition (C_2), the following esti- mate is obtained from Eq. (3.1.13),

$$\begin{aligned} \|y^{(0)} - x^*\| &\leq \frac{\int_0^1 \phi((1-\theta)\|x^{(0)} - x^*\|)\,d\theta}{1 - \phi_0(\|x^{(0)} - x^*\|)}\|x^{(0)} - x^*\| \\ &\leq \chi_1(\|x^{(0)} - x^*\|)\|x^{(0)} - x^*\| < \|x^{(0)} - x^*\| < \rho^*, \end{aligned} \tag{3.1.14}$$

Thus, Eq. (3.1.9) is valid for k = 0, proving that the iteration y(0) is a member of B(x*, ρ∔). Moreover, considering the estimate (3.1.12) for u = y(0), we may verify that the linear operator F′(y(0)) is invertible.

Now, rewrite the second step of method (3.1.1) for k = 0 as

$$\begin{aligned} x^{(1)} - x^* &= y^{(0)} - x^* - F'(y^{(0)})^{-1}F(y^{(0)}) - Q(x^{(0)}, y^{(0)})F'(y^{(0)})^{-1}F(y^{(0)}) \\ &= F'(y^{(0)})^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}\left[F'(y^{(0)}) - F'(x^* + \theta(y^{(0)} - x^*))\right](y^{(0)} - x^*)d\theta \\ &\quad - Q(x^{(0)}, y^{(0)})F'(y^{(0)})^{-1}F'(x^*)F'(x^*)^{-1}F(y^{(0)}). \end{aligned} \tag{3.1.15}$$

Then, by using

$$\begin{aligned} \|F'(x^*)^{-1}F(y^{(0)})\| &\leq \int_0^1 \|F'(x^*)^{-1}F'(x^* + \theta(y^{(0)} - x^*))\| \|y^{(0)} - x^*\| d\theta \\ &\leq \int_0^1 \|I + F'(x^*)^{-1}(F'(x^* + \theta(y^{(0)} - x^*)) - F'(x^*))\| \|y^{(0)} - x^*\| d\theta \\ &\leq \left[1 + \int_0^1 \phi_0(\theta\|y^{(0)} - x^*\|)d\theta\right] \|y^{(0)} - x^*\|, \end{aligned}$$

The estimate is then obtained by applying Eqs. (3.1.15) to the data from (C2) to (C4), as well as Eqs. (3.1.4), (3.1.12) (for u = y (0)), and (3.1.14).

$$\begin{aligned} \|x^{(1)} - x^*\| &\leq \Bigg[\frac{\int_0^1 \phi((1-\theta)\|y^{(0)} - x^*\|)d\theta}{1 - \phi_0(\|y^{(0)} - x^*\|)} + \psi(\|x^{(0)} - x^*\|, \|y^{(0)} - x^*\|) \\ &\qquad \times \frac{1 + \int_0^1 \phi_0(\theta\|y^{(0)} - x^*\|)d\theta}{1 - \phi_0(\|y^{(0)} - x^*\|}\Bigg] \|y^{(0)} - x^*\| \\ &\leq \chi_2(\|x^{(0)} - x^*\|)\|x^{(0)} - x^*\| < \|x^{(0)} - x^*\| < \rho^*. \end{aligned}$$

Since iteration x(1) is included in B(x*, ρ*), Equation (3.1.10) is true for k = 0. When the words When the induction process on 'k' is completed for the estimates in (3.1.9) to (3.1.10), the inferred approximations usually use x(k), y(k), and x(k+1) instead of x(0), y(0), and x(1), respectively.

Moreover, in view of the following estimate,

$$\|x^{(k+1)} - x^*\| < \beta\|x^{(k)} - x^*\| < \rho^*,$$

Where $\beta = \chi_2(||x^{(0)} - x^*||) \in [0, 1)$, i it eventually follows that $x^{(k+1)} \in B(x^*, \rho^*)$ for all k ∈ N, and consequently $\lim_{k\to\infty} x^{(k)} = x^*$.

The following statement may therefore be used to assert that the answer is unique:

Proposition : Assume that:

1. For any positive integer ρ¯, there exists a solution x* ∈ B(x*, ρ¯) ⊆ Ω such that F(x) = 0.
2. On B(x*, ρ¯), conditions (C1) and (C2) are true.
3. The condition that ρ˜ ≥ ρ is met.

$$\int_0^1 \phi_0(\theta\tilde{\rho})d\theta < 1.$$

Let Ω_3 = B[x*, ρ˜] ∩ **Ω**. In such case, x* is the only area where the equation F(x) = 0 may be solved Ω_3.

Proof: Suppose $x^{**} \in \Omega_3$ solves $F(x) = \mathbf{0}$. Define the linear operator,

$$\Phi = \int_0^1 F'(x^* + \theta(x^{**} - x^*))d\theta.$$

By applying the given conditions (i)–(iii), the following is obtained,

$$\begin{aligned} \|F'(x^*)^{-1}(\Phi - F'(x^*))\| &\le \int_0^1 \phi_0(\theta\|x^{**} - x^*\|)d\theta \\ &\le \int_0^1 \phi_0(\theta\tilde{\rho})d\theta < 1. \end{aligned}$$

Hence, $\Phi{-1} \in L\,(B_2, B_1)$ x* = x** is an obvious consequence considering

$$\Phi(x^{**} - x^*) = F(x^{**}) - F(x^*) = \mathbf{0}.$$

This proves the result.

We might easily select ρ¯ = ρ* if Proposition made use of the criteria (C3) and (C4), but they are not.

In addition to the uniqueness findings for the considered IM, which were not previously given, the technique in Additionally, the approximate computational errors on x(k) - x* are provided in this section this is worth noting (3.1.2).

3.1.2 Semilocal convergence analysis

Our next step is to construct the iterative semilocal convergence analysis approach given by Eq. (3.1.2) in this section. Let S = [0, ∞) and assume the following requirements to be true in order to begin:

1. **There exists a function $\xi_0 : S \rightarrow S$** that functions $\xi_0(t) - 1$ has the lowest zero and is both non-decreasing and continuous $r_0 \in S \setminus \{0\}$. Let $\bar{S} = [0, r_0)$.
2. **There exists functions, $\xi : \bar{S} \rightarrow S$** and $\tau : \bar{S} \times \bar{S} \rightarrow S$, both of which are on-going and not declining.

Next, provide definitions to the scalar sequences {α(k)} and {H (k)} for all k = 0, 1, 2,..., as long as α(0) = 0, β (0) = μ for some μ ≥ 0, and

$$\alpha^{(k+1)} = \beta^{(k)} + \frac{1+\tau(\alpha^{(k)},\beta^{(k)})}{1-\xi_0(\beta^{(k)})}\left[\int_0^1 \xi(\theta(\beta^{(k)}-\alpha^{(k)}))d\theta\right](\beta^{(k)}-\alpha^{(k)}), \tag{3.1.16}$$

$$\beta^{(k+1)} = \alpha^{(k+1)} + \frac{1}{1-\xi_0(\alpha^{(k+1)})}\Bigg[\int_0^1 \xi(\theta(\alpha^{(k+1)}-\alpha^{(k)}))d\theta\,(\alpha^{(k+1)}-\alpha^{(k)}) + (1+\xi_0(\alpha^{(k)}))(\alpha^{(k+1)}-\beta^{(k)})\Bigg]. \tag{3.1.17}$$

The following is evidence that the sequences specified by Eqs. (3.1.16) and (3.1.17) are the most significant for the iterate sequence generated using approach (3.1.1). The definition of majorizing sequence is provided by Definition. The fact that the sequence {γ(k)} does not decrease is clearly stated in this formulation. Furthermore, if it exceeds the boundaries by $\bar{\gamma} \geq 0$, it must converge to a value between 0 and γ*. The sequence {x(k)} will converge to a specific value of x* if the given normed linear space is exhaustive. It follows that

$$\|x^* - x^{(k)}\| \leq \gamma^* - \gamma^{(k)}, \quad \text{for each } k = 0, 1, 2, \ldots.$$

Iterative algorithm developers must keep this concept in mind while developing semilocal convergence analysis. To set the stage for the major Assuming the sequences given by Eqs. converge, the following lemma gives a generic condition for this convergence. (4.2.16)-(4.2.17).

Lemma : For all k = 0, 1, 2, . . ., assume that:

$$\xi_0(\alpha^{(k)}) < 1, \qquad (3.1.18)$$

$$\xi_0(\beta^{(k)}) < 1, \qquad (3.1.19)$$

$$\alpha^{(k)} \leq \bar{r}_0, \quad \textit{for some } \bar{r}_0 \leq r_0. \qquad (3.1.20)$$

Next, the monotonic convergence of the sequence {α(k)} to its unique minimum upper limit ϱ*∈ [0, r¯0] is achieved.

The proof. The sequence {α(k)} is clearly non-decreasing and confined above by r¯0, according to the conditions provided by Eqs. (3.1.18) - (3.1.20) and the way it is expressed. As a result, it will face its clearly defined upper limit α in due time.

This method's semilocal convergence analysis relies on the following conditions:

Assume that:

(H$_1$): There exists a point $x^{(0)}$ ∈ Ω such that $F'(x^{(0)})^{-1}$ ∈ L (B_2, B_1) and

$$\|F'(x^{(0)})^{-1}F(x^{(0)})\| \leq \mu.$$

(H2): For each x ∈ Ω,

$$\|F'(x^{(0)})^{-1}(F'(x) - F'(x^{(0)}))\| \leq \xi_0(\|x - x^{(0)}\|).$$

Let Ω4 = B(x(0), r0) ∩ Ω.

(H3): For each x, y ∈ Ω4

$$\|F'(x^{(0)})^{-1}(F'(x)-F'(y))\| \leq \xi(\|x-y\|),$$

And

$$\|Q(x,y)\| \leq \tau(\|x-x^{(0)}\|, \|y-x^{(0)}\|).$$

(H4): Conditions of Lemma 4.2.1 hold.

(H5): B[x(0), α∗] ⊂ Ω.

The semilocal analysis is now shown below using the criteria (H1)-(H5) and the created notation.

Equation : Make it seem as if (H1)–(H5) are really true. According to equation (3.1.1), the technique yields sequences that meet the following assertions for every k = 0, 1, 2,...:

$$x^{(k)} \in B(x^{(0)}, \alpha^*), \tag{3.1.21}$$

$$\|y^{(k)} - x^{(k)}\| \leq \beta^{(k)} - \alpha^{(k)}, \tag{3.1.22}$$

$$\|x^{(k+1)} - y^{(k)}\| \leq \alpha^{(k+1)} - \beta^{(k)}, \tag{4.2.23}$$

and further, $\lim_{k\to\infty} x^{(k)} = x^* \in B[x^{(0)}, \alpha^*]$, *such that* $F(x^*) = \mathbf{0}$.

Proof: In order to prove the statements in Eqs. (3.1.21) - (3.1.23), mathematical induction will be used. Equation (3.1.21) seems to be valid by definition for k= 0. By using the assumptions (H1) and (H2), we are now free to examine any arbitrary u ∈ B(x(0), α*).

$$\|F'(x^{(0)})^{-1}(F'(u)-F'(x^{(0)}))\| \leq \xi_0(\|u-x^{(0)}\|) \leq \xi_0(\alpha^*) < 1,$$

And therefore, as a consequence of Invertible operators and the Banach lemma,

$$\|F'(u)^{-1}F'(x^{(0)})\| \leq \frac{1}{1-\xi_0(\|u-x^{(0)}\|)}. \tag{3.1.24}$$

The iteration y(0) exists and is well-defined according to the first step of the technique (4.2.1), which is determined by condition (H1) and Eq. (3.1.24) (for u = x(0)). Up next, we have

$$\|y^{(0)}-x^{(0)}\| = \|F'(x^{(0)})^{-1}F(x^{(0)})\| \leq \mu = \beta^{(0)}-\alpha^{(0)} < \alpha^{*}.$$

Accordingly, the estimate (3.1.22) is valid for k = 0, and the iteration y(0) belongs to B(x(0), α::). Taking into consideration the following estimate,

$$\begin{aligned} F(y^{(0)}) &= F(y^{(0)}) - F(x^{(0)}) - F'(x^{(0)})(y^{(0)}-x^{(0)}) \\ &= \int_0^1 [F'(x^{(0)}+\theta(y^{(0)}-x^{(0)})) - F'(x^{(0)})]d\theta\ (y^{(0)}-x^{(0)}), \end{aligned}$$

The following is also derived by using the criteria (H2) and (H3) in conjunction with Eq. (3.1.24) (for u = y(0)):

$$\begin{aligned} \|F'(y^{(0)})^{-1}F(y^{(0)})\| &= \|F'(y^{(0)})^{-1}F'(x^{(0)})F'(x^{(0)})^{-1}F(y^{(0)})\| \\ &\leq \|F'(y^{(0)})^{-1}F'(x^{(0)})\|\|F'(x^{(0)})^{-1}\int_0^1 [F'(x^{(0)}+\theta(y^{(0)}-x^{(0)})) \\ &\qquad - F'(x^{(0)})]d\theta\ (y^{(0)}-x^{(0)})\| \\ &\leq \frac{1}{1-\xi_0(\|y^{(0)}-x^{(0)}\|)}\left[\int_0^1 \xi(\theta\|y^{(0)}-x^{(0)}\|)d\theta\right]\|y^{(0)}-x^{(0)}\|. \end{aligned} \tag{3.1.25}$$

Hence, the second step of (3.1.1) produces the desired result when the sequence {α(k)} is defined in conjunction with Eq. (3.1.25).

$$\begin{aligned} \|x^{(1)}-y^{(0)}\| &= \|F'(y^{(0)})^{-1}F(y^{(0)}) + Q(x^{(0)},y^{(0)})F'(y^{(0)})^{-1}F(y^{(0)})\| \\ &\leq \frac{1+\tau(\|x^{(0)}-x^{(0)}\|,\|y^{(0)}-x^{(0)}\|)}{1-\xi_0(\|y^{(0)}-x^{(0)}\|)}\left[\int_0^1 \xi(\theta\|y^{(0)}-x^{(0)}\|)d\theta\right]\|y^{(0)}-x^{(0)}\| \\ &\leq \alpha^{(1)}-\beta^{(0)}. \end{aligned}$$

Consequently,

$$\begin{aligned} \|x^{(1)}-x^{(0)}\| &= \|x^{(1)}-y^{(0)}\| + \|y^{(0)}-x^{(0)}\| \\ &\leq \alpha^{(1)}-\beta^{(0)}+\beta^{(0)}-\alpha^{(0)} < \alpha^{*}. \end{aligned}$$

That is, the estimate (3.1.23) is valid for k = 0, and the iteration x(1) belongs to B(x(0), α*). Ultimately, the proof must be made that y(1) is a member of B(x(0), α*). Right now, let's use this phrase,

$$\begin{aligned} F(x^{(1)}) &= F(x^{(1)}) - F(x^{(0)}) - F'(x^{(0)})(x^{(1)} - x^{(0)}) + F'(x^{(0)})(x^{(1)} - y^{(0)}) \\ &= \int_0^1 \left[F'(x^{(0)} + \theta(x^{(1)} - x^{(0)})) - F'(x^{(0)})\right] d\theta \, (x^{(1)} - x^{(0)}) + F'(x^{(0)})(x^{(1)} - y^{(0)}), \end{aligned}$$

And further using the conditions (H2) and (H3) In addition to defining the sequence {\ (k)}, the first step of method (3.1.1) for k = 1 produces,

$$\begin{aligned} \|y^{(1)} - x^{(1)}\| &\le \|F'(x^{(1)})^{-1} F'(x^{(0)})\| \|F'(x^{(0)})^{-1} F(x^{(1)})\| \\ &\le \frac{1}{1 - \xi_0(\|x^{(1)} - x^{(0)}\|)} \Big[\int_0^1 \xi(\theta \|x^{(1)} - x^{(0)}\|) d\theta \, \|x^{(1)} - x^{(0)}\| \\ &\qquad + (1 + \xi_0(\|x^{(0)} - x^{(0)}\|)) \|x^{(1)} - y^{(0)}\| \Big] \\ &\le \beta^{(1)} - \alpha^{(1)}. \end{aligned}$$

Consequently

$$\begin{aligned} \|y^{(1)} - x^{(0)}\| &= \|y^{(1)} - x^{(1)}\| + \|x^{(1)} - x^{(0)}\| \\ &\le \beta^{(1)} - \alpha^{(1)} + \alpha^{(1)} - \alpha^{(0)} < \alpha^*. \end{aligned}$$

The iteration y(1) belongs to B(x(0), α*), and the provided estimate (3.1.22) is valid for k = 1. To complete the induction process for statements (3.1.21)-(3.1.23), restate the prior estimations while appropriately substituting x(0), y(0), and x(1) with x(k), y(k), and x(k+1), respectively. The estimate is also available for any k.

$$\|F'(x^{(0)})^{-1} F(x^{(k+1)})\| \le \int_0^1 \xi(\theta \alpha^{(k+1)} - \alpha^{(k)}) d\theta \, (\alpha^{(k+1)} - \alpha^{(k)}) + (1 + \xi_0(\alpha^{(k)}))(\alpha^{(k+1)} - \beta^{(k)}). \tag{3.1.26}$$

Since the series {α(k)} is convergent according to condition (H4) A basic element x∏ ∈ B[x(0), α∏] exists such that limk→∞ x(k) = x*, and the sequence {x(k)} is fundamental in Banach space. The fact that F(x*) = 0 may be inferred from the fact that k might approach infinity in the estimate (3.1.26) and the continuity of operator F. This finding of uniqueness is derived from the following statement:

Proposition : Assume that:

1. Equation $F(x) = 0$ has a solution $x* \in B(x(0), \bar{\alpha}) \subseteq \Omega$ for some $\bar{\alpha} > 0$ and $F'(x(0))-1 \in L(B2, B1)$.
2. Conditions (H1) and (H2) hold.
3. There exists $\tilde{\alpha} \geq \bar{\alpha}$ such that

$$\int_0^1 \xi_0(\theta\tilde{\alpha}+(1-\theta)\bar{\alpha})d\theta < 1. \tag{3.1.27}$$

The set $\Omega 5$ can be defined as the set $B[x(0), \tilde{\alpha}] \cap \wp$. After that, in the interval $\Omega 5$, $x^*$is the only solution to the equation $F(x) = 0$. The evidence. Let the operator Φ be defined as in Proposition 4.2.1 and think of x^{**}as an element of $\Omega 5$ with $F(x^{**}) = 0$. Equation (3.1.27) is used in conjunction with the assumptions (H1) and (H2), to

$$\begin{aligned}\|F'(x^{(0)})^{-1}(\Phi - F'(x^{(0)}))\| &\leq \int_0^1 \xi_0(\theta\|x^{**}-x^{(0)}\|+(1-\theta)\|x^{*}-x^{(0)}\|)d\theta \\ &\leq \int_0^1 \xi_0(\theta\tilde{\alpha}+(1-\theta)\bar{\alpha})d\theta < 1.\end{aligned}$$

Thus, $\Phi-1 \in L(B2, B1)$ and we immediately deduce that $x* = x**$.

4. CONCLUSION

Improving computational mathematics relies heavily on creating and studying novel iterative methods for numerically solving nonlinear equations. When compared to more conventional approaches like Newton-Raphson or Secant methods, these newer methods strive to be more efficient computationally while simultaneously increasing convergence speed and accuracy. Many new higher-order iterative techniques have been suggested in the literature recently; these schemes aim to improve resilience and decrease the amount of iterations needed for convergence by combining parameters, memory effects, and hybrid strategies. The major goal is to deal with various nonlinear issues, such as those involving singularities, ill-conditioned functions, or numerous roots. The practical application of novel methods in engineering, physics, and scientific computing is ensured by often constructing them with a balance between the order of convergence and processing cost each iteration. Rigid proofs back up theoretical convergence assessments, while numerical tests, frequently benchmarking against standard test functions, confirm performance. In certain cases, the results show that these new strategies are far more effective than the traditional ones.

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