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**A COMPARATIVE ANALYSIS ON DOMINATING  
COLOUR AND BOUNDED COMPONENT  
TRANSVERSALS OF GRAPHS**

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# A Comparative Analysis on Dominating Colour and Bounded Component Transversals of Graphs

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**Abstract** – The subject of this paper is a relaxation of proper graph colorings - bounded monochromatic component colorings (BMC colorings). A vertex-coloring of a graph is called a BMC coloring if every color-class induces monochromatic components containing at most a certain bounded number of vertices. A proper coloring for instance is a BMC coloring in which every color-class induces monochromatic components of order one. We investigate three different aspects of BMC colorings.

We investigate extremal graph theoretic problems of BMC colorings. For certain families of graphs we determine bounds for the smallest monochromatic component, order  $C$ , the critical component order, such that every graph contained in this family accommodates for a BMC coloring with respect to  $C$ . We determine bounds for the critical component order  $C$  for graphs with a bounded maximum degree: Every graph of maximum degree at most three admits a BMC 2-coloring with one color-class inducing monochromatic components of order one and the other color-class inducing monochromatic components of order at most 22; and every graph of maximum degree at most five admits a BMC 2-coloring inducing monochromatic components of order at most 1908 in each of the two color-classes. Additionally we restrict, the graphs to being planar and show that every maximal planar graph (a triangulation) with maximum degree  $\Delta$  and containing at most,  $d$  vertices of odd degree admits a BMC 3-coloring inducing monochromatic components of order at most  $2\Delta d$ .

To almost all proofs related to BMC colorings there is a common denominator: bounded component transversals of multipartite graphs. Thus we devote the first part of this paper to these transversals, proving both extremal and algorithmic results. We investigate the smallest number of vertices  $n(\Delta')$  that still guarantees the existence of (an efficient algorithm for finding) a transversal inducing bounded components in every multipartite graph with partite sets of order at least  $n(\Delta')$  and maximum degree at most  $\Delta'$ . We further emphasize the importance of transversals inducing bounded components with an application to the Linear Arboricity Conjecture.

## INTRODUCTION

**Graph Colorings** : A graph  $G$  is a set  $V(G)$  of vertices and a set  $E(G)$  of edges, each connecting a pair of vertices, its endpoints. We say that the two endpoints of an edge are adjacent. A drawing that shows the well known Petersen Graph can be found in Figure 1(a).

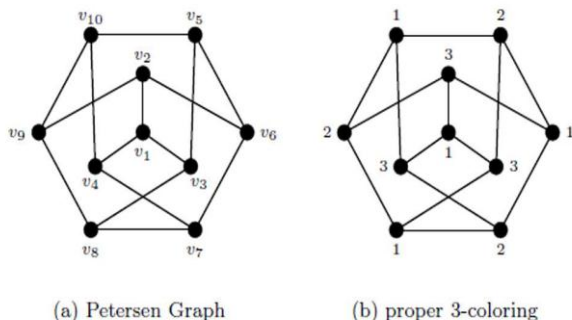


Figure 1: The Petersen Graph and a proper 3-coloring of its vertices. A proper  $k$ -coloring of the vertices of a graph  $G$  is an assignment, of  $k$  colors (often the integers  $1, \dots, k$ ) so that no two adjacent vertices get the same color, see for instance Figure 1(b) for a proper 3-coloring of the Petersen Graph. The set of vertices receiving color  $j$  is a color-class and induces a graph with no edges, i.e., it is an independent set of  $G$ . So, a proper  $k$ -coloring of the vertices of  $G$  is simply a partition of  $V(G)$  into  $k$  independent sets (compare Figure 1(b)).

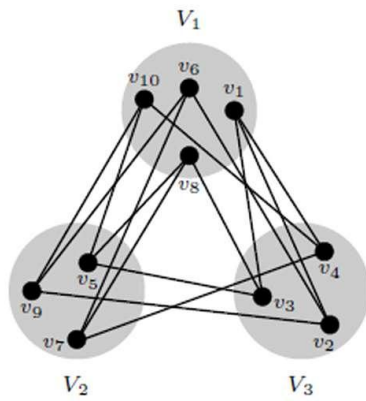


Figure 2: A partition of the Petersen Graph into three independent sets.

The chromatic number  $\chi(G)$  of a graph  $G$  is the minimum  $k$  for which there is a  $k$ -coloring of  $G$ . One straightforward reason for a graph to have large chromatic number is the containment of a large clique (i.e., a large complete subgraph). Obviously if a graph  $G$  contains a clique of order  $k$ , then  $\chi(G) \geq k$ .

Similarly for every subgraph  $H \subseteq G$  it holds that  $\chi(H) \leq \chi(G)$ . Since the Petersen Graph contains odd cycles, and an odd cycle requires at least, three colors in any proper coloring, the chromatic number of the Petersen Graph is at least three. The following proposition states a relation between the maximum degree  $\Delta(G)$  of a graph  $G$  and its chromatic number.

**Bounded Component Transversals :** Let us recall that a transversal of a multipartite graph is a subset, of its vertices containing exactly one vertex from each partite set. The transversal graph is the graph induced by the vertices of the transversal. For an integer  $f \geq 1$  we call a transversal an  $f$ -bc transversal if the largest, component of the transversal graph contains at most  $f$  vertices.

The main goal of this part of the paper is to guarantee the existence of bounded component transversals for multipartite graphs that have large partite sets, with respect to their maximum degree. Moreover we want to elaborate on algorithms that even find such bounded component transversals. It seems natural that for some fixed maximum degree, the more vertices a multipartite graph contains in each of its partite sets, the more freedom is given to choose a transversal, possibly inducing only small components in the transversal graph. We have seen in Proposition 0.2 that the (list-)chromatic number of a graph can easily be upper bounded by its maximum degree, independent of the number of vertices contained in the graph. Let us for a moment focus on independent transversals, i.e., transversals inducing an independent, set.

Let  $G$  be a multipartite graph with maximum degree  $\Delta$  for which the number of vertices in each of its partite sets is lower-bounded by some value  $p(\Delta)$  depending only on  $\Delta$ . In contrast to the greedy algorithm for proper (list-)colorings, one can observe that a greedy algorithm for choosing an independent, transversal of  $G$  - choose a vertex  $t_i$  from  $K$  that is not adjacent to any of the earlier chosen vertices  $t_j$  in  $G$ ,  $j < i$  is likely to fail. This algorithm can get stuck in a partial transversal  $t_1, \dots, t_k$  such that every vertex of  $V_{k+i}$  is adjacent to a vertex of  $\{t_1, \dots, t_k\}$ , with  $k < m$ .

Bounded component, transversals have proved to be very applicable in many areas of graph theory. In this paper we show an application of bounded component, transversals in order to get one tiny step closer to a proof of the Linear Arboricity Conjecture.

**Bounded Monochromatic Component Colorings** Sometimes the number of colors available to color a graph is less than its chromatic number. Therefore one is forced to relax the properness condition and to find a good approximation of its properness. Another good reason to introduce relaxations of proper colorings is that in some theoretical or practical situations a small deviation from proper is still acceptable, while the problem could become tractable, or in certain problems the use of the full strength of proper coloring is an "overkill". Often a weaker concept, suffices and provides better overall results.

## BOUNDED COMPONENT TRANSVERSALS

Let  $G$  be a multipartite graph with  $V(G) = V_1 \cup \dots \cup V_m$ . A transversal  $T$  of  $G$  is a subset of the vertices in  $G$  containing exactly one vertex from each partite set  $V_i$ . An  $f$ -bc transversal of a graph with a vertex partition is a transversal  $T$  in which each connected component of the subgraph induced by  $T$  (the transversal graph) has at most  $f$  vertices. Let  $\Delta_f(m, n)$  denote the smallest integer  $\Delta$  such that there exists an

$f$ -bc transversal of a graph  $G$  with maximum degree  $\Delta$  and parts of size  $n$  and with no  $f$ -bc transversal. We define  $\Delta_f(n) = \min_{m \in \mathbb{N}} \Delta_f(m, n)$  in case we do not want to restrict the number of parts in the graphs under consideration and let  $\Delta_f$  denote  $\lim_{n \rightarrow \infty} \Delta_f(n)/n$ . It is not hard to check that this limit always exists.

**Independent Transversals :** Historically the investigation of  $\Delta_f(m, n)$  started with 1-bc transversals, subsequently called independent transversals. Independent transversals and in particular the determination of the

number  $\Delta_1(m, n)$ , for  $m \geq 2$  and  $n \geq 1$  received a lot of attention. In a series of works  $\Delta_1(m, n)$  has been completely determined.

Theorem 1.3. For integers  $m \geq 2$  and  $n \geq 1$  the following holds,

$$\Delta_1(m, n) = \begin{cases} \lceil \frac{(m-1)n}{2(m-2)} \rceil, & \text{if } m \text{ is odd, and} \\ \lceil \frac{mn}{2(m-1)} \rceil, & \text{if } m \text{ is even.} \end{cases}$$

The upper bound for all  $m$  and  $n$  has been proved by Szabo and Tardos for even  $m$  in and in is has been shown that this construction is also optimal for odd in (adding one more partite set). We give here a simplified construction showing  $n/2 < \Delta_1(m, n) \leq \lceil n/2 \rceil + 1$  for

$m - 2 \geq N$  only. This yields a tight bound if  $n$  is even. Let  $G$  be the graph in Figure 3 (a vertex represents a whole class of vertices). That is,  $G$  is an  $r$ -partite graph with partite sets of size  $n$ ,  $m > n$  and each partite set  $V_i, i \in [m-2]$  is partitioned into two almost equally sized

classes  $V_{i,1}$  and  $V_{i,2}$  (i.e.,  $|V_{i,1}| = \lfloor n/2 \rfloor$  and  $|V_{i,2}| = \lceil n/2 \rceil$ ). Let every vertex in part  $V_{i,2}$  be adjacent to every vertex in  $V_{i+1,1}$ , with  $i \in [m-3]$  and let every vertex in  $V_{m-2,2}$  be adjacent, to every vertex in  $V_{i,1}$ . We assume that, the two parts  $V_{m-1}$  and  $V_m$  each contain

$n \leq m - 2$  vertices

$v_{m-1,1}, v_{m-1,2}, \dots, v_{m-1,n}$  and

$v_{m,1}, v_{m,2}, \dots, v_{m,n}$ , respectively. Further let

every vertex of  $V_{i,j}$  be adjacent to  $v_{m+1-j,i}$ ,  $i \in [n]$  and  $j \in \{1, 2\}$

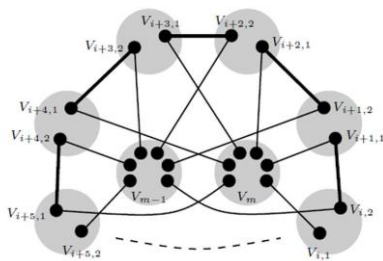


Figure 3: An  $r$ -partite graph  $G$  without an independent transversal.

Matching Transversals : Let us finally restrict to 2-bc transversals which we subsequently want to call a matching transversals. According to Theorem every multipartite graph  $G$  with partite sets of order  $n$  and  $\Delta(G) \leq \lfloor 2n/3 \rfloor$  contains a matching transversal. On the other hand according to Proposition there arc multipartite graphs with partite sets of order  $n$  and  $\Delta(G) = n + 1$  without a matching transversal. We think that these graphs are in some sense optimal.

Theorem. Every multipartite graph  $G$  with partite sets of order at least two and with maximum degree  $\Delta(G) \leq 2$  contains a match ing transversal  $T$ , i.e.,  $\Delta_2(2) = 2$ . Moreover there is a linear-time (in the order of  $G$ ) algorithm that finds  $T$ . We will

investigate  $\Delta_f(r, n)$  for matching transversals ( $f = 2$ ). We can construct graphs such that the following holds.

## ALGORITHMIC ASPECTS

Throughout this paper we restrict ourselves to independent, transversals. We call the running-time of an algorithm with input graph  $G$  polynomial if it is

polynomial in the order of  $G$ , assuming that  $\Delta(G)$  is constant. Let  $G$  be a multipartite graph with partite sets containing at least  $n$  vertices. The goal of this paper is to derive a polynomial-time algorithm that finds an independent transversal of  $G$ . In order to do so we are forced to strengthen the condition on the

partite set sizes from  $n \geq 2\Delta(G)$  (for which  $G$  is guaranteed to contain an independent

transversal according to Theorem 1.1)

to  $n = \Omega(\Delta(G)^3)$ . Note here that Alon mentions that a deterministic polynomial-time algorithm exists that finds an independent transversal of  $G$  even if

only  $n \geq C\Delta(G)$  holds,  $G$  being a large constant. We believe that in a future work our results can be combined with the techniques of Alon to obtain an algorithm that finds an independent transversal in every multipartite graph  $G$  with parts

containing at least  $C'\Delta(G)$  many vertices, for some constant  $C' \geq 2$  that is much smaller than the constant  $G$  implicitly given in.

## BOUNDED MONOCHROMATIC COMPONENT COLORINGS

Let us recall the definition of BMC colorings. We say that a  $k$ -fc-coloring of a graph is  $(C_1, C_2, \dots, C_k)$ -BMC if every



monochromatic component induced by the vertices of the  $i$ th color-class is of order at most  $C_i$ , for  $i \in [k]$ . Note that a  $(1, \dots, l)$ -BMC  $k$ -coloring corresponds to a proper  $\wedge$ -coloring. We mainly deal with the two most natural cases of BMC  $k$ -colorings. We say C-symmetric BMC  $k$ -coloring (also C-sbmc  $k$ -coloring) when  $C_i = C_j$ , for  $i, j \in [k]$ . Similarly we say C-asymmetric BMC  $(k, l)$ -coloring (also C-ABMC  $(k, l)$ -coloring) when  $C_i = 1$ , for  $1 \leq i \leq k$  and  $C_j = C$ , for  $k+1 \leq j \leq k+l$ . For 2-colorings refer to a C-ABMC  $(1, l)$ -coloring also by a C-ABMC 2-coloring.

In this part of the paper we investigate BMC colorings of graphs with bounded maximum degree. Symmetric BMC colorings were first studied by Kleinberg, Motwani, Raghavan, and Venkatasubramanian by in. In this paper we focus on BMC 2-colorings. Symmetric BMC 2-colorings have been studied by Alon, Ding, Oporowski, and Vertigan and implicitly, even earlier, by Thomassen who resolved the problem for the line graph of 3-regular graphs initiated by Akiyama and Chvatal. Asymmetric BMC 2-colorings were first introduced in a joint paper with Tibor Szabo.

In this paper we investigate BMC  $k$ -colorings of bounded degree (planar) graphs,  $k > 2$ .

Also Haxell, Szabo, and Tardos in, Linial, Matousek, Shchect, and Tardos in, and Matousek and Prfvetiv in investigated (symmetric) BMC colorings.

Independently Andrews and Jacobson, Harary and Jones, and Cowen, Cowen, and Woodall introduced and investigated the concept of improper colorings over various families of graphs. A  $\wedge$ -coloring is called  $\wedge$ -improper if none of the at most  $k$  colors induces a monochromatic component containing vertices of degree larger than  $\wedge$ . Hence in an improper coloring the amount of error is measured in terms of the maximum degree of monochromatic components rather than in terms of their order. Several papers on the topic have since appeared; in particular, two papers, by Eaton and Hull and Skrczkovski, have extended the work of Cowen et al. to a list-coloring variant of improper colorings.

Linial and Saks studied low diameter graph decompositions, where the quality of the coloring is measured by the diameter of the monochromatic components. Their goal was to color graphs with as few colors as possible such that each monochromatic connected component has a small diameter.

Haxell, Pikhurko, and Thomason study the fragmentability of graphs introduced by Edwards and Farr, in particular for bounded degree graphs. A graph is called  $(\alpha, f)$ -fragmentable if one can remove a

fraction of the vertices and end up with components of order at most  $\wedge$ . For comparison, in a C-ABMC 2-coloring one must remove an independent set and end up with small components.

The so-called relaxed chromatic number (sometimes also called generalized chromatic number) was introduced by Weaver and West. They used "relaxation" in a much more general sense than us, requiring that each color-class is the member of a given family of graphs.

## SBMC COLORINGS OF PLANAR GRAPHS

In this section we investigate symmetric BMC colorings of planar graphs. Due to the Four Color Theorem we know that every planar graph  $G$  admits a proper 4-coloring, hence  $\text{sbmc}_4(G) = 1$ . On the other hand there are planar graphs which cannot be properly colored with only three colors. Moreover for a planar graph  $G$  the determination whether  $G$  is properly 3-colorable is NP-complete, see Theorem 0.1.

Nevertheless the following simple characterization of triangulations that are properly 3-colorable has been shown by Haxwood.

**Theorem :** The vertices of a triangulation  $G$  are properly 3-colorable if and only if  $G$  is Eulerian.

In contrast to the long and computer aided proof of the Four Color Theorem, and even its simplification by Robertson, Sanders, Seymour, and Thomas, there is a rather compact proof by Cowen, Cowen, and Woodall not assuming the truth of the Four Color

**Theorem** that  $\text{sbmc}_4(G) \leq 2$  for every planar graph  $G$ .

As mentioned earlier amongst the many results of, the authors show the existence of planar graphs  $H_1$  with arbitrarily many vertices and with maximum degree at most six such

that  $\text{sbmc}_2(H_1) = \Omega(\sqrt{|V(H_1)|})$ . Linial, Matousek, Shchect, and Tardos study bounded monochromatic component, colorings of minor closed classes of graphs. They show for instance that for

every planar graph  $G$ ,  $\text{sbmc}_2(G) = O(|V(G)|^{2/3})$  and they construct planar graphs

$H_2$  with  $\text{sbmc}_2(H_2) = \Omega(|V(H_2)|^{2/3})$ .

For 3-colorings Kleinberg, Motwani, Raghavan, and Venkatasubramanian by in construct planar graphs

such that  $\text{sbmc}_3(H_3) = \Omega(|V(H_3)|^{1/3})$ .

These graphs have large maximum degree, that is,  $\Delta(H_3) = \Omega(|V(H_3)|)$ . Motivated by this they ask for

the following which we want to formulate as a conjecture.

## CONCLUSION

Theory of domination and graph colouring theory are two important as well as fastest growing areas in combinatorics. A number of variations of domination have been introduced by several authors. In this sequence, Benedict et al. introduced a new variation in domination, namely, chromatic transversal domination which involves both domination and colouring. Also partitioning the vertex set  $V$  of a graph  $G$  into subsets of  $V$  having some property is one direction of research in graph theory. For instance, one such partition is domatic partition which is a partition of  $V$  into dominating sets. Analogously, we here demand each set in the partition of  $V$  to have the property of chromatic transversal domination instead of just domination and call this partition a chromatic transversal domatic partition. Further, the maximum order of such partition is called the chromatic transversal domatic number which is denoted by  $dct(G)$ .

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