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STUDY ON APPLICATION OF LAGRANGE'S THEOREM IN COSETS

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Study on Application of Lagrange's Theorem in Cosets

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Abstract – Lagrange's Theorem, one of the most important results in finite group theory, states that the order of a subgroup must divide the order of the group. This theorem provides a powerful tool for analyzing finite groups; it gives us an idea of exactly what type of subgroups we might expect a finite group to possess. Central to understanding Lagrange's Theorem is the notion of a coset.

Key Words: Theorem, Subgroup, Analyzing Finite Groups

INTRODUCTION

A coset is what we get when we take a subgroup and shift it (either on the left or on the right). The best way to think about cosets is that they are shifted subgroups, or translated subgroups.

Note g lies in both gH and Hg , since $g = ge = eg$. Typically $gH \neq Hg$. When G is abelian, though, left and right cosets of a subgroup by a common element are the same thing. When an abelian group operation is written additively, an H -coset should be written as $g + H$, which is the same as $H + g$.

Example 1.2. In the additive group Z , with subgroup mZ , the mZ -coset of a is $a + mZ$.

This is just a congruence class modulo m .

Example 1.3. In the group R_+ , with subgroup $H = f_+g$, the H -coset of x is $xH = fx; \square xg$. This is $\setminus x$ up to sign."

Example 1.4. When $G = S_3$, and $H = f(1); (12)g$, the table below lists the left H -cosets and right H -cosets of every element of the group. Compute a few of them for non-identity elements to satisfy yourself that you understand how they are found.

g	gH	Hg
(1)	{(1), (12)}	{(1), (12)}
(12)	{(1), (12)}	{(1), (12)}
(13)	{(13), (123)}	{(13), (132)}
(23)	{(23), (132)}	{(23), (123)}
(123)	{(13), (123)}	{(23), (123)}
(132)	{(23), (132)}	{(13), (132)}

Notice first of all that cosets are usually not subgroups (some do not even contain the identity). Also, since $(13)H \neq H(13)$, a particular element can have different left and right H -cosets. Since $(13)H = (123)H$, different elements can have the same left H -coset. (You have already seen this happen with congruences: $14 + 3Z = 2 + 3Z$, since $14 \equiv 2 \pmod{3}$.)

Properties of cosets

We will generally focus our attention on left cosets of a subgroup. Proofs of the corresponding properties of right cosets will be completely analogous, and can be worked out by the interested reader.

Since $g = ge$ lies in gH , every element of G lies in some left H -coset, namely the left coset defined by the element itself. (Take a look at Example 1.4, where (13) lies in (13)H.) Similarly, $g \in Hg$ since $g = eg$.

A subgroup is always a left and a right coset of itself: $H = eH = He$. (This is saying nothing other than the obvious fact that if we multiply all elements of a subgroup by the identity, on either the left or the right, we get nothing new.) What is more important to recognize is that we can have $gH = H$ (or $Hg = H$) even when g is not the identity. For instance, in the additive group Z , $10 + 5Z = 5Z$. All this is saying is that if we shift the multiples of 5 by 10, we get back the multiples of 5. Isn't that obvious? In fact, the only way we can have $a + 5Z = 5Z$ is if a is a multiple of 5, i.e., if $a \in 5Z$.

For a subgroup H of a group G , and $g \in G$, when does gH equals H ?

Theorem 4.1. For $g \in G$, $gH = H$ if and only if $g \in H$.

Proof. Since $g = g e \in gH$, having $gH = H$ certainly requires $g \in H$.

Now we need to show that if $g \in H$, then $gH = H$. We prove $gH = H$ by showing each is a subset of the other. Since $g \in H$, $gh \in H$ for any $h \in H$, so $gH \subseteq H$. To see $H \subseteq gH$, note $h = g(g^{-1}h)$ and that $g^{-1}h$ is in H (since $g \in H$).

Joseph-Louis Lagrange (1736{1813), born in Turin, Italy, was of French and Italian descent. His talent for mathematics became apparent at an early age. Leonhard Euler recognized Lagrange's abilities when Lagrange, who was only 19, communicated to Euler some work that he had done in the calculus of variations. That year he was also named a professor at the Royal Artillery School in Turin. At the age of 23 he joined the Berlin Academy. Frederick the Great had written to Lagrange proclaiming that the "greatest king in Europe" should have the "greatest mathematician in Europe" at his court. For 20 years Lagrange held the position vacated by his mentor, Euler. His works include contributions to number theory, group theory, physics and mechanics, the calculus of variations, the theory of equations, and differential equations. Along with Laplace and Lavoisier, Lagrange was one of the people responsible for designing the metric system. During his life Lagrange profoundly incensed the development of mathematics, leaving much to the next generation of mathematicians in the form of examples and new problems to be solved.

REVIEW OF LITERATURE:

In ancient Greece, three classic problems were posed. These problems are geometric in nature and involve straightedge-and-compass constructions from what is now high school geometry; that is, we are allowed to use only a straightedge and compass to solve them. The problems can be stated as follows.

1. Given an arbitrary angle, can one trisect the angle into three equal sub-angles using only a straightedge and compass?
2. Given an arbitrary circle, can one construct a square with the same area using only a straightedge and compass?
3. Given a cube, can one construct the edge of another cube having twice the volume of the original? Again, we are only allowed to use a straightedge and compass to do the construction.

After puzzling mathematicians for over two thousand years, each of these constructions was finally shown to be impossible. We will use the theory of fields to provide a proof that the solutions do not exist. It is quite remarkable that the long-sought solution to each of these three geometric problems came from abstract algebra.

First, let us determine more specifically what we mean by a straightedge and compass, and also examine the nature of these problems in a bit more

The index and Lagrange's theorem

For any integer $m \neq 0$, the number of cosets of $m\mathbb{Z}$ in $\mathbb{Z}$ is $|m|$. This gives us an

interesting way to think about the meaning of $|m|$, other than its definition as $|m|$ made

positive." Passing from $\mathbb{Z}$ to other groups, counting the number of cosets of a subgroup

gives a useful numerical invariant.

Definition 5.1. Let H be a subgroup of a group G . The index of H in G is the number of

left cosets of H in G . This number, which is a positive integer or 1, is denoted $[G : H]$.

Concretely, the index of a subgroup tells us how many times we have to translate the

subgroup around (on the left) to cover the whole group.

Example 5.2. Since $H = \{(1); (12)\}$ has three left cosets in S_3 , by Example 4.4, $[S_3 : H] =$

3.

Example 5.3. The subgroup $H = \{1; s\}$ of D_4 has four left cosets:

$H; rH = \{r; rs\}; r^2H = \{r^2; r^2s\}; r^3H = \{r^3; r^3s\};$

The index of H in D_4 is 4.

Example 5.4. For a positive integer m , $[\mathbb{Z} : m\mathbb{Z}] = m$, since $0; 1; \dots; m-1$ are a complete set of coset representatives of $m\mathbb{Z}$ in $\mathbb{Z}$.

Example 5.5. What is the index of $15\mathbb{Z}$ inside $3\mathbb{Z}$? (Not inside $\mathbb{Z}$, but $3\mathbb{Z}$.) Modulo 15, a multiple of 3 is congruent to $0; 3; 6; 9$, or 12 . That is, we have the disjoint union $3\mathbb{Z} = 15\mathbb{Z} \cup (3 + 15\mathbb{Z}) \cup (6 + 15\mathbb{Z}) \cup (9 + 15\mathbb{Z}) \cup (12 + 15\mathbb{Z})$. Thus $[3\mathbb{Z} : 15\mathbb{Z}] = 5$.

MATERIAL AND METHOD:

Lagrange's Theorem, one of the most important results in finite group theory, states that the order of a subgroup must divide the order of the group. This theorem provides a powerful tool for analyzing finite groups; it gives us an idea of exactly what type of subgroups we might expect a finite group to possess. Central to understanding Lagrange's Theorem is the notion of a coset.

Cosets

Let G be a group and H a subgroup of G . Define a left coset of H with representative $g \in G$ to be the set

$$gH = \{gh : h \in H\}$$

Right cosets can be defined similarly by

$$Hg = \{fhg : h \in H\}$$

If left and right cosets coincide or if it is clear from the context to which type of coset that we are referring, we will use the word coset without specifying left or right.

Example 1. Let H be the subgroup of Z_6 consisting of the elements 0 and

3. The cosets are

$$0 + H = 3 + H = \{0, 3\}$$

$$1 + H = 4 + H = \{1, 4\}$$

$$2 + H = 5 + H = \{2, 5\}$$

We will always write the cosets of subgroups of Z and Z_n with the additive notation we have used for cosets here. In a commutative group, left and right cosets are always identical.

Example 2. Let H be the subgroup of S_3 defined by the permutations

$f(1); (123); (132)g$. The left cosets of H are

$$(1)H = (123)H = (132)H = \{(1); (123); (132)\}$$

$$(12)H = (13)H = (23)H = \{(12); (13); (23)\}$$

The right cosets of H are exactly the same as the left cosets:

$$H(1) = H(123) = H(132) = \{(1); (123); (132)\}$$

$$H(12) = H(13) = H(23) = \{(12); (13); (23)\}$$

It is not always the case that a left coset is the same as a right coset. Let

K be the subgroup of S_3 defined by the permutations $f(1); (12)g$. Then the

left cosets of K are

$$(1)K = (12)K = \{(1); (12)\}$$

$$(13)K = (123)K = \{(13); (123)\}$$

$$(23)K = (132)K = \{(23); (132)\}$$

however, the right cosets of K are

$$K(1) = K(12) = \{(1); (12)\}$$

$$K(13) = K(132) = \{(13); (132)\}$$

$$K(23) = K(123) = \{(23); (123)\}$$

The following lemma is quite useful when dealing with cosets. (We leave its proof as an exercise.)

Lemma 6.1 Let H be a subgroup of a group G and suppose that $g_1, g_2 \in G$.

The following conditions are equivalent.

$$1. g_1H = g_2H;$$

$$2. Hg_1 = Hg_2;$$

$$3. g_1H \subseteq g_2H;$$

$$4. g_2 \in g_1H;$$

$$5. g \in 1$$

$$1. g_2 \in H.$$

CONCLUSION:

In all of our examples the cosets of a subgroup H partition the larger group G .

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