Journal of Advances in Science and Technology

Vol. VI, Issue No. XII, February-2014, ISSN 2230-9659

AN EVALUATION ON NUMEROUS APPLICATIONS OF LINEAR DIFFERENTIAL OPERATORS

AN INTERNATIONALLY INDEXED PEER REVIEWED & REFEREED JOURNAL

Journal of Advances in Science and Technology
Vol. VI, Issue No. XII, February-2014, ISSN 2230-9659

An Evaluation on Numerous Applications of Linear Differential Operators

Abid Mushtaq[1] Dr. R. S. Singh[2]

[1]Research Scholar, Jodhpur National University, Jodhpur

[2]Deputy Director, K. L.S. Institute of Engineering and Technology, Bijnor

Abstract – At the Edinburgh congress 12 years ago Gârding gave a general survey of the theory of linear partial differential operators. I shall take his lecture as my starting point and try to give some idea of the later development. Naturally it is necessary to concentrate on a few topics and ignore others which are as interesting. I shall not try to list the omissions but wish to specify the limitation to questions concerning the existence and structure of solutions of differential equations with constant, C^{∞}or analytic coefficients.

In this paper we will begin to take a more sophisticated approach to differential equations. We will define, with some care, the notion of a linear differential operator, and explore the analogy between such operators and matrices.

---------------------------♦-----------------------------

INTRODUCTION

Differential equations seem to be well suited as models for systems. Thus an understanding of differential equations is at least as important as an understanding of matrix equations. In Section 1.5 we inverted matrices and solved matrix equations. In this paper we explore the analogous inversion and solution process for linear differential equations.

Because of the presence of boundary conditions, the process of inverting a differential operator is somewhat more complex than the analogous matrix inversion. The notation ordinarily used for the study of differential equations is designed for easy handling of boundary conditions rather than for understanding of differential operators. As a consequence, the concept of the inverse of a differential operator is not widely understood among engineers. The approach we use in this paper is one that draws a strong analogy between linear differential equations and matrix equations, thereby placing both these types of models in the same conceptual framework.

The key concept is the Green's function. It plays the same role for a linear differential equation as does the inverse matrix for a matrix equation. There are both practical and theoretical reasons for examining the process of inverting differential operators. The inverse (or integral form) of a differential equation displays explicitly the input-output relationship of the system. Furthermore, integral operators are computationally and theoretically less troublesome than differential operators; for example, differentiation emphasizes data errors, whereas integration averages them.

Consequently, the theoretical justification for applying many of the computational procedures to differential systems is based on the inverse (or integral) description of the system. Finally, the application of the optimization techniques to differential systems often depends upon the prior determination of the integral forms of the systems.

One of the reasons that matrix equations are widely used is that we have a practical, automatable scheme, Gaussian elimination, for inverting a matrix or solving a matrix equation. It is also possible to invert certain types of differential equations by computer automation. The greatest progress in understanding and automation has been made for linear, constant-coefficient differential equations with initial conditions.

LINEAR DIFFERENTIAL OPERATORS WITH CONSTANT COEFFICIENTS

The general linear ODE of order n is

$$y^{(n)} + p_1(x)y^{(n-1)} + \ldots + p_n(x)y \;=\; q(x). \qquad (1)$$

From now on we will consider only the case where (1) has constant coefficients. This type of ODE can be written as

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y \; = \; q(x)\; ; \qquad (2)$$

using the differentiation operator D, we can write (2) in the form

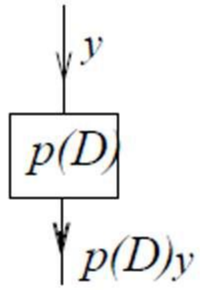
$$(D^n + a_1 D^{n-1} + \ldots + a_n)\, y \; = \; q(x) \qquad (3)$$

or more simply, where $p(D)\, y \; = \; q(x)$,

$$p(D) = D^n + a_1 D^{n-1} + \ldots + a_n \; . \qquad (4)$$

We call p(D) a polynomial differential operator with constant coefficients. We

think of the formal polynomial p(D) as operating on a function y(x), converting- it into another function; it is like a black box, in which the function y(x) goes in. and p{D)y comes out.



Our main goal in this section of the Notes is to develop methods for finding particular solutions to the ODE (2) when q(x) has a special form: an exponential, sine or cosine, x^k, or a product of these. (The function q(x) can also be a sum of such special functions.) These are the most important functions for the standard applications.

The reason for introducing the polynomial operator p(D) is that this allows us to use polynomial algebra to help find the particular solutions. The rest of this paper of the Notes will illustrate this. Throughout, we let

$$p(D) = D^n + a_1 D^{n-1} + \ldots + a_n \; , \qquad (4)$$

a_i constants.

DIFFERENTIAL OPERATORS

In this section, we study differential equations and their associated differential operators. Only properties of very simple differential equations can be proved by working with their solutions, e.g., linear differential equations with constant coefficients that form a nilpotent matrix. Differential Operators. More complicated differential equations need a different approach, because their solutions may not fall into decidable classes of arithmetic, are not computable, or may not even exist in closed form. As a proof technique for advanced differential equations, we have introduced differential invariants. Differential invariants turn the following intuition into a formally sound proof procedure. If the vector field of the differential equation always points into a direction where the differential invariant F, which is a logical formula, is becoming "more true", then the system will always stay safe if it initially starts safe. This principle can be understood in a simple but formally sound way in the logic dC. Differential in-variants have been introduced in, and later refined to a procedure that computes differential invariants in a fixed-point loop. Instead of our original presentation, which was based on differential algebra, total deriva-tives, and differential substitution, we take a differential operator approach here. Both views are fruitful and closely related.

Definition 1 (Lie differential operator). Let $x' = \theta$ be the differential equation system $x_1' = \theta_1, \ldots, x_n' = \theta_n$ in vectorial notation. The (Lie) differential operator belonging to $x' = \theta$ is the operator $\theta \cdot \nabla$ defined as

$$\theta \cdot \nabla \stackrel{\text{def}}{\equiv} \sum_{i=1}^{n} \theta_i \frac{\partial}{\partial x_i} = \theta_1 \frac{\partial}{\partial x_1} + \cdots + \theta_n \frac{\partial}{\partial x_n} \qquad (1)$$

The $\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\}$ are partial derivative operators, but can be considered as a basis of the tangent space at x of the manifold on which $x' = \theta$ is defined. The result of applying the differential operator $\theta \cdot \nabla$ to a differentiate function is

$$(\theta \cdot \nabla) f = \sum_{i=1}^{n} \theta_i \frac{\partial f}{\partial x_i} = \theta_1 \frac{\partial f}{\partial x_1} + \cdots + \theta_n \frac{\partial f}{\partial x_n}$$

The differential operator lifts conjunctively to logical formulas F:

$$(\theta \cdot \nabla) F \stackrel{\text{def}}{\equiv} \bigwedge_{(b \sim c) \text{ in } F} ((\theta \cdot \nabla) b \sim (\theta \cdot \nabla) c)$$

This conjunction is over all atomic subformulas $b \sim c$ of F for any operator

$\sim \in \{=, \geq, >, \leq, <\}$. In this definition, we assume that formulas use dualities like $\neg(a \geq b) \equiv a < b$ to avoid negations and the operator $\neq$ is handled in a special way; se previous work for a discussion. The functions and terms in / and F need to be sufficiently smooth for the partial derivatives to be defined and enjoy useful

Abid Mushtaq[1] Dr. R. S. Singh[2]

properties like commutativity of $\frac{\partial}{\partial x}$ land $\frac{\partial}{\partial y}$. This is the case for polynomials, which are arbitrarily smooth (C^∞).

EQUATIONS WITH ANALYTIC COEFFICIENTS

Hyperfunctions.- In the study of differential operators with C00 coefficients it is natural to work with Schwartz distributions which form the largest class on which all such operators are defined. However, when the coefficients are real analytic it is possible to work within the larger frame of Sato hyperfunctions. During the past few years much work has been done along such lines which has given many results parallel to those for Schwartz distributions. We must content ourselves here with referring to the survey by Schapira and the lecture by M. Sato in these proceedings.

Uniformization - A study of the Cauchy problem with data on a hypersurface which is partly characteristic was initiated by Leray. He found that the solution ramifies around the variety generated by the bicharacteristics passing through the characteristic points of the initial surface. A detailed analysis was given by Gärding, Kotake and Leray in the case of linear systems. Later Choquet-Burhat has simplified the proofs and extended the general result to non-linear equations.

CONCLUSION

Differential invariants are a natural induction principle for differential equations. The structure of general differential invariants has been studied previously. Here, we took a differential operator view and have studied the case of equational differential invariants in more detail. We have related equational differential invariants to Lie's seminal work and subsequent results about Lie groups. We have shown how the resulting equivalence characterization of invariant equations on open domains can be used, carefully illustrate surprising challenges in invariant generation, explain why they exist, and show with which techniques they can be overcome. We have studied the structure of invariant functions and invariant equations, their relation, and have shown that, in the presence of differential cuts, the invariant equations and provable invariant equations form a chain of differential ideals and that their varieties are generated by a single invariant.

Finally, we relate differential invariants to partial differential equations and explain how the inverse characteristic method reduces the problem of educational differential invariant generation to that of solving partial differential equations.

REFERENCES

- Alur, R., Courcoubetis, C., Halbwachs, N., Henzinger, T.A., Ho, P.H., Nicollin, X., Olivero, A., Sifakis, J., Yovine, S.: The algorithmic analysis of hybrid systems. Theor. Comput. Sci. 138(1), 3{34 (1995)
- Evans, L.C.: Partial Di_erential Equations, Graduate Studies in Mathematics, vol. 19. AMS, 2nd edn. (2010)
- Forsythe, George E. and Wolfgang R. Wasow, Finite-Difference Methods for Partial Differential Equations, Wiley, New York, 1960.
- Grigor'ev, D.Y.: Complexity of quanti_er elimination in the theory of ordinary di_erential equations. In: Davenport, J.H. (ed.) EUROCAL. LNCS, vol. 378, pp. 11{25. Springer (1987)
- K. G. ANDERSSON. — Propagation of analyticity of solutions of partial differential equations with constant coefficients, Ark. for Matematik, 8 (1970).
- Stakgold, Ivar, Boundary Value Problems of Mathematical Physics, Volume I, Macmillan, New York, 1968.