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A Research about Different Modeling and Application of Higher Order Derivatives

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Abstract – Modeling of high quality surfaces is the core of geometric modeling. Such models are used in many computer-aided design and computer graphics applications. Irregular behavior of higher-order differential parameters of the surface (e.g. curvature variation) may lead to aesthetic or physical imperfections. In this work, we consider methods for constructing surfaces with high degree of smoothness.

This is an overview of recent research of the authors on the application of variational methods with higher–order derivatives in image processing. We focus on gray-valued and matrix-valued images and deal with a purely discrete setting. We show that regularization methods with second–order derivatives can be successfully applied to the denoising of gray–value images.

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INTRODUCTION

High-order PDEs (fourth- and sixth-order in particular) arise in many geometric modeling operations requiring surface optimization: blending, hole-filling, curve network interpolation and interactive surface editing.

In many cases, these PDEs are derived from functionals involving second- and third-order quantities (for example, curvature and curvature variation) in a variational setting. However, in many other cases geometric PDEs not derived from functional optimization can yield equally good results but with less complex formulation.

Several formulations may be used in the discretization of high-order functionals. In one direction, sufficiently high-order basis functions are used to represent the surface. In this case, all higher-order derivatives can be computed point-wise. An advantage of this is the need for fewer discretization points to describe a smooth surface, similar to the limited number of control points needed to specify a patch. This way, one can avoid the need for highly refined meshes for the smooth surface approximation. Another advantage is the straightforward computation of the derivatives of functionals, as there exists a closed form describing the surface. Furthermore, convergence guarantees are easier to provide and faster convergence rates can be achieved.

BACKGROUND

High-quality free-form surface design requires formalizing the notion of surface quality, choosing equations for surface evolution resulting in the improvement of numerical quality criteria, and efficient techniques for solving these equations. In this section we review related work on each of these aspects of free-form surface design.

Surface quality is typically evaluated using surface interrogation methods. Such methods visualize and quantitatively characterize the quality of different types of characteristic lines on surfaces: isophotes, principal curvature lines and reflection lines.

These methods usually are not directly applied to *construct* or *improve* surfaces. Instead, a separate fairness functional or a flow equation is used to obtain an improved surface. Then interrogation methods are used to evaluate its quality, and, if necessary, adjust the way the surface is obtained.

PDEs and functionals, as all necessary derivatives can be computed explicitly point-wise. However, in many applications it is essential to be able to deal with high-resolution meshes directly, both in the context of interrogation and surface construction and optimization.

HIGHER–ORDER DERIVATIVES IN IMAGE PROCESSING

In recent years mathematical methods from optimization theory, harmonic analysis, stochastics or partial differential equations were successfully applied in digital image processing, while conversely image processing tasks have led to interesting mathematical questions. In this paper, we restrict our attention to applications of variational methods in conjunction with higher–order derivatives in image processing. In a couple of papers, these techniques have proved to be useful for scalar-valued images, vector-valued images and tensor-valued images. In this paper, we are only interested in scalar- and matrix–valued images, more precisely in the denoising of gray–value images and matrix fields. Vector-valued images are for example colored images or optical flow fields, see Fig. 1 (middle). One of the authors has used higher–order regularization methods for the simultaneous estimation and decomposition of optical flows. Matrix-valued data have gained significant importance in recent years, e.g., in diffusion tensor magnetic resonance imaging (DT-MRI). Here, every image pixel corresponds

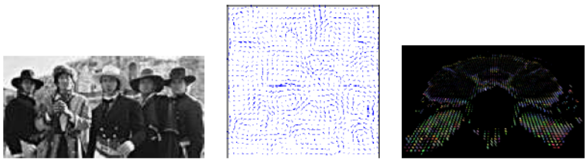

Fig. 1. Gray-value image of the battle at the Alamo in San Antonio (left), vector-valued image of an optical flow field (middle), matrix-valued image of a DT-MRI slice (right).

to a symmetric positive definite matrix A which can be visualized as the ellipsoid

$$\{x \in \mathbb{R}^3 : x^{\mathrm{T}} A^{-2} x = 1\}.$$

The lengths of the axes of the ellipsoid are the eigenvalues of A and the ellipsoid illustrates the direction of the diffusion of water molecules, see Fig. 1 (right).

A well-established method for the denoising of a scalar-valued image u from a given image f degraded by white Gaussian noise consists in calculating

$$\underset{u}{\operatorname{argmin}} \int_\Omega (f-u)^2 + \alpha\, \Phi(|\nabla u|^2)\, dx dy$$

with a regularization parameter $\alpha > 0$ and an increasing function $\Phi : [0, \infty] \to \mathbb{R}$ in the penalizing term. The first summand encourages simi-larity between the restored image and the original one, while the second term rewards smoothness. For the straightforward choice $\Phi(s) = s^2$, the penalizing term coincides with the H^1 norm of *u.* The corresponding minimizcr becomes too smooth at edges. The frequently applied ROF-modcl introduced by Rudin. Oshcr and Fatemi with $\Phi(s^2) := \sqrt{s^2} = |s|$ preserves sharp edges, but leads to the so-called *staircasing effect.* We will see that one way to overcome both artifacts is to use higher-order derivatives in the functional.

HIGHER ORDER DERIVATIVES OF LYAPUNOV FUNCTIONS

Consider the dynamical system

$$\dot{x} = f(x), \quad (1)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ has an equilibrium point at the origin (i.e., f (0) = 0), and satisfies the standard assumptions for existence and uniqueness of solutions; see e.g. Chap. 3]. By higher order derivatives of a Lyapunov function $V(x) : \mathbb{R}^n \to \mathbb{R}$ we mean the time derivatives of V along the trajectories of (1) given by $\dot{V}(x) = \langle \frac{\partial V(x)}{\partial x}, f(x) \rangle, \ddot{V}(x) = \langle \frac{\partial \dot{V}(x)}{\partial x}, f(x) \rangle, \ldots, V^{(m)}(x) = \langle \frac{\partial V^{(m-1)}(x)}{\partial x}, f(x) \rangle$. In, Butz showed that existence of a three times continuously differentiable Lyapunov function V(x) satisfying

$$\tau_2 \dddot{V}(x) + \tau_1 \ddot{V}(x) + \dot{V}(x) < 0 \quad (2)$$

for all $x \neq 0$ and for some nonnegative scalars τ_1, τ_2

implies global asymptotic stability of the origin of (1). Note that unlike the standard condition $\dot{V}(x) < 0$, condition (2) is not jointly convex in the scalars τ_i and the parameters of the Lyapunov function V(x). Therefore, computational techniques based on convex optimization cannot be used to search for a Lyapunov function satisfying (2). In, Heinen and Vidyasagar adapted the condition of Butz to establish a result on boundedness of the trajectories. More recently, Meigoli and Nikravesh have generalized the result of Butz to derivatives of higher order and to the case of time-varying systems. A simplified version of their result that is most relevant for our purposes deals with a differential inequality of the type

$$V^{(m)}(x) + \tau_{m-1} V^{(m-1)}(x) + \cdots + \tau_1 \dot{V}(x) < 0. \quad (3)$$

It is shown in that if the corresponding characteristic polynomial

$$p(s) = s^m + \tau_{m-1} s^{m-1} + \cdots + \tau_1 s$$

is Hurwitz (and some additional standard assumptions hold), then the inequality in (3) proves global asymptotic stability. It is later shown in that this condition can be weakened to $p(s)$ having nonnegative coefficients. We will show that no matter what types

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of conditions on V(x) and the scalars $\tau_{m-1}, \ldots, \tau_1$ are placed, if the system is globally asymptotically stable and the inequality (3) holds (which is in particular the case if inequality (3) is used to establish global asymptotic stability), then we can explicitly extract a standard Lyapunov function from it. This will follow as a corollary of the following simple and general fact.

Theorem 1.1: Consider a system $\dot{x} = f(x)$ that is known to be globally asymptotically stable. Suppose there exists a continuously differentiable function *W (x)* whose derivative $\dot{W}(x)$ along the trajectories is negative definite and satisfies W(0) = 0. Then, W(x) must be positive definite.

Proof: Assume by contradiction that there exists a nonzero point $\bar{x} \in \mathbb{R}^n$ such that $W(\bar{x}) \leq 0$. We evaluate the Lyapunov function W(x) along the trajectory of the system starting from the initial condition $\bar{x}$. The value of the Lyapunov function is nonpositive to begin with and will strictly decrease because $\dot{W}(x) < 0$. Therefore, the value of the Lyapunov function can never become zero. On the other hand, since we know that the vector field is globally asymptotically stable, trajectories of the system must all go to the origin, where we have W(0) = 0. This gives us a contradiction.

CONCLUSION

Modeling of high quality surfaces is the core of geometric modeling. Such models are used in many computer-aided design and computer graphics applications. Irregular behavior of higher-order differential parameters of the surface (e.g. curvature variation) may lead to aesthetic or physical imperfections. In this work, we presented three techniques for constructing surfaces with high degree of smoothness.

As an alternative, we described a discrete-geometric construction for a simple and efficient method for discretizing reflection line based functionals on meshes and demonstrated how these functionals can be used in an interactive system to optimize the shape of reflective surfaces.

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