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A STUDY ON SEQUENTIAL LIMIT OF A SEQUENCE ON CLOSED SUBSET OF R

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A Study on Sequential Limit of a Sequence on Closed Subset of R

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Abstract *– Limits can be defined in any metric or topological space, but are usually first encountered in the real numbers.*

x ***the limit of the sequence*** (x_n) ***if the following condition holds:***

For each real number $\epsilon > 0$***, there exists a natural number*** N ***such that, for every natural number*** $n > N$***, we have*** $|x_n - x| < \epsilon$***.***

In other words, for every measure of closeness ϵ***, the sequence's terms are eventually that close to the limit. The sequence*** (x_n) ***is said to converge to or tend to the limit*** x***, written*** $x_n \to x$ ***or*** $\lim_{n\to\infty} x_n = x$***.***

If a sequence converges to some limit, then it is convergent; otherwise it is divergent.

The limit of a sequence is the value that the terms of a sequence "tend to". If such a limit exists, the sequence is called convergent. A sequence which does not converge is said to be divergent. The limit of a sequence is said to be the fundamental notion on which the whole of analysis ultimately rests.

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INTRODUCTION

Limits of sequences behave well with respect to the usual arithmetic operations. If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$, $a_n b_n \to ab$ and, if neither *b* nor any b_n is zero, $a_n / b_n \to a/b$.

For any continuous function *f*, if $x_n \to x$ then $f(x_n) \to f(x)$. In fact, any real-valued function *f* is continuous if and only if it preserves the limits of sequences (hough this is not necessarily true when using more general notions of continuity.

Examples

- If $x_n = c$ for some constant *c*, then $x_n \to c$. *Proof*: choose $N = 1$. We have that, for every $n > N$, $|x_n - c| = 0 < \epsilon$.
- If $x_n = 1/n$, then $x_n \to 0$. *Proof*: choose $N = \left\lfloor \frac{1}{\epsilon} \right\rfloor$ (the floor function). We have that, for every $n > N$, $|x_n - 0| \le x_{N+1} = \frac{1}{\lfloor 1/\epsilon \rfloor + 1} < \epsilon$.
- If $x_n = 1/n$ when n is even, and $x_n = 1/n^2$ when n is odd, then $x_n \to 0$. (The fact that $x_{n+1} > x_n$ whenever n is odd is irrelevant.)
- Given any real number, one may easily construct a sequence that converges to that number by taking decimal approximations. For example, the sequence $0.3, 0.33, 0.333, 0.3333, \ldots$ converges to $1/3$. Note that the decimal representation $0.3333\ldots$ is the *limit* of the previous sequence, defined by $0.3333\ldots \triangleq \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{10^i}$.
- Finding C might sometimes be non-intuitive, like $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$, the number *e*. In these cases, one common approach is to find upper and lower bounds for the limit of the sequence (e.g., proving that $2.71 < e < 2.72$).

Some other important properties of limits of real sequences include the following.

- The limit of a sequence is unique.

- $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n$
- $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n)$
- $\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ provided $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$
- If $a_n \leq b_n$ for all n greater than some N, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$
- (Squeeze Theorem) If $a_n \leq c_n \leq b_n$ for all $n > N$, and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$, then $\lim_{n\to\infty} c_n = L$.
- If a sequence is bounded and monotonic then it is convergent.
- A sequence is convergent if and only if every subsequence is convergent.

These properties are extensively used to prove limits without the need to directly use the cumbersome formal definition. Once proven that $1/n \to 0$ it becomes easy to show that $\frac{a}{b+c/n} \to \frac{a}{b}$, ($b \neq 0$), using the properties above.

LIMIT OF SEQUENCES ON CLOSED SUBSET

The terminology and notation of convergence is also used to describe sequences whose terms become very large. A sequence (x_n) is said to tend to infinity, written $x_n \to \infty$ or $\lim_{n\to\infty} x_n = \infty$ if, for every *K*, there is an *N* such that, for every $n \geq N$, $x_n > K$; that is, the sequence terms are eventually larger than any fixed *K*. Similarly, $x_n \to -\infty$ if, for every *K*, there is an *N* such that, for every $n \geq N$, $x_n < K$.

1. The limit of a convergent sequence is unique.

2. Every convergent sequence is bounded.
This is a quite interesting result since it implies that if a sequence is not bounded, it is therefore divergent. For example, the sequence is not bounded, therefore it is divergent.

3. Any bounded increasing (or decreasing) sequence is convergent.

Note that if the sequence is increasing (resp. decreasing), then the limit is the least-upper bound (resp. greatest-lower bound) of the numbers x_n, for $n = 1, 2, \cdots$.

4. If the sequences $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are convergent and α and β are two arbitrary real numbers, then the new sequence $\{\alpha x_n + \beta y_n\}_{n\geq 1}$ is convergent. Moreover, we have

$$\lim_{n\to\infty}\left(\alpha x_n + \beta y_n\right) = \alpha\lim_{n\to\infty} x_n + \beta\lim_{n\to\infty} y_n.$$

It is also true that the sequence $\{x_n \cdot y_n\}_{n\geq 1}$ is convergent and

$$\lim_{n\to\infty}\left(x_n \cdot y_n\right) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n.$$

5. If the sequence $\{x_n\}_{n\geq 1}$ is convergent and $\lim_{n\to\infty} x_n \neq 0$ and $x_n \neq 0$ for any $n \geq 1$, then the sequence $\left\{\frac{1}{x_n}\right\}_{n\geq 1}$ is convergent. Moreover, we have

$$\lim_{n\to\infty}\frac{1}{x_n} = \frac{1}{\lim_{n\to\infty} x_n}.$$

The following examples will be useful to familiarize yourself with limit of sequences.

Example: Show that for any number *a* such that 0 < *a* <1, we have

$$\lim_{n\to\infty} a^n = 0.$$

Answer: Since 0 < *a* <1, then the sequence $\{a^n\}$ is obviously decreasing and bounded; hence it is convergent. Write

$$\lim_{n\to\infty} a^n = L.$$

We need to show that *L*=0. We have

$$\lim_{n\to\infty} a^{n+1} = L,$$

since the sequence $\{a^{n+1}\}$ is a tail of the sequence $\{a^n\}$; hence they have the same limit. But,

$$a^{n+1} = aa^n$$

using the previous properties, we get

$$\lim_{n\to\infty} a^{n+1} = a\cdot\lim_{n\to\infty} a^n,$$

which implies

$$L = a \cdot L.$$

Since $a \neq 0$, then we must have L=0.

One may wonder, what happened to the sequence $\{a^n\}$ if a > 1? It is divergent since it is not bounded. This follows from

$$a^n = \frac{1}{\left(\frac{1}{a}\right)^n}$$

and

$$\lim_{n\to\infty}\left(\frac{1}{a}\right)^n = 0.$$

Remark: Note that it is possible to talk about a sequence of numbers which converges to $\pm\infty$. Of course, we do reserve the word convergent to sequences which converges to a number; $\pm\infty$ is not a number. The following shows the process:

The sequence $\{x_n\}_{n\geq 1}$ converges to ∞(or, to $-\infty$), if and only if, for any real number M > 0, there exists an integer $N \geq 1$, such that for any $n \geq N$, we have $x_n \geq M$ (or $x_n \leq -M$).

In particular, if $x_n \to 0$ as $n \longrightarrow \infty$ and $x_n > 0$ for any $n \geq 1$, then we have

$$\lim_{n\to\infty}\frac{1}{x_n} = +\infty$$

A sequence which converges to $\pm\infty$ is obviously not bounded.
For example, we have

$$\lim_{n\to\infty} a^n = +\infty$$

for any a > 1.

A sequence $\{xn\}n \in N \subset X$ is said to **converge towards** $x0 \in X$ if for any ε>0 there is a natural number $n\varepsilon$ with the property that $xn \in B\varepsilon(x0)$ for all $n \geq n\varepsilon$:

Lim $n\to\infty xn = x0 \Leftrightarrow \text{def} \forall \varepsilon>0 \exists\ n\ \varepsilon \in N; xn \in B\varepsilon(x0)$ for $n \geq n\varepsilon$.

We then say that xn tends to $x0$ as n tends to infinity, written

$xn\to x0$(as $n\to\infty$), or $xn\to n\to\infty x0$.

The point x_0 is called the **limit** of the sequence $\{xn\}n \in N$.

Sequential limits are zero limits for the distance function .Since $\{dX(xn,x0)\}n \in N$ is a sequence in R it is easily verified that

$xn\to x0 \Leftrightarrow dX(xn,x0)\to 0$.

We say that $f(x)$ **converges to** $y0$ **in** Y **as** x **converges to** $x0$ **in** X if for any ε>0 there exists δ>0 such that $f(x) \in B\varepsilon(y0)$ when $x \in B\delta(x0)$:

Lim $x\to x0 f(x)=y0 \Leftrightarrow \text{def} \forall\varepsilon>0 \exists\delta>0;[dX(x,x0)<\delta \Rightarrow dY(f(x),y0)<\varepsilon]$.

Equivalent ways of writing this are

$f(x)\to y0$ as $x\to x0$, and $f(x)\to x\to x0 y0$.

A function f satisfying this is said to be **continuous** at the point $x0$. It is continuous on a set D if it continuous at all points $x0 \in D$, and simply continuous if its continuous on all of its domain.

limit of a function is a fundamental concept in calculus and analysis concerning the behavior of that function near a particular input.

Formal definitions, first devised in the early 19th century, are given below. Informally, a function f assigns an output $f(x)$ to every input x. The function has a limit L at an input p if $f(x)$ is "close" to L whenever x is "close" to p. In other words, $f(x)$ becomes closer and closer to L as x moves closer and closer to p. More specifically, when f is applied to each input *sufficiently* close to p, the result is an output value that is *arbitrarily* close to L. If the inputs "close" to p are taken to values that are very different, the limit is said to *not exist*.

The notion of a limit has many applications in modern calculus. In particular, the many definitions of continuity employ the limit: roughly, a function is continuous if all of its limits agree with the values of the function. It also appears in the definition of the derivative: in the calculus of one variable, this is the limiting value of the slope of secant lines to the graph of a function.

To say that

$$\lim_{x \to p} f(x) = L,$$

means that $f(x)$ can be made as close as desired to L by making x close enough, but not equal, to p.

The following definitions (known as (ε, δ)-definitions) are the generally accepted ones for the limit of a function in various contexts.

Functions on the real line

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined on the real line and $p, L \in \mathbb{R}$. It is said the limit of f as x approaches p is L and written

$$\lim_{x \to p} f(x) = L,$$

if the following property holds:

- For every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all real x, $0 < |x - p| < \delta$ implies $|f(x) - L| < \varepsilon$.

Note that the value of the limit does not depend on the value of $f(p)$, nor even that p be in the domain of f.

A more general definition applies for functions defined on subsets of the real line. Let (a, b) be an open interval in R, and p a point of (a, b). Let f be a real-valued function defined on at least all of $(a, b) \setminus \{p\}$. It is then said that the limit of f as x approaches p is L if, for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that $0 < |x - p| < \delta$ and $x \in (a, b)$ implies $|f(x) - L| < \varepsilon$. Note that the limit does not depend on $f(p)$ being well-defined.

The letters ε and δ can be understood as "error" and "distance", and in fact Cauchy used ε as an abbreviation for "error" in some of his work these terms, the error (ε) in the measurement of the value at the limit can be made as small as desired by reducing the distance (δ) to the limit point. As discussed below this definition also works for functions in a more general context. The idea that δ and ε represent distances helps suggest these generalizations.

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