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NUMERICAL ANALYSIS FOR SINGULARITY PERTURBED DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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Numerical Analysis for Singularity Perturbed Differential Equations and Its Applications

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Abstract – In this talk, I will discuss the role of numerical analysis in the design of numerical algorithms to approximately solve certain classes of singularly perturbed differential equations. The solutions of singularly perturbed differential equation have narrow layer regions in the domain, where the solution exhibits steep gradients. Classical numerical methods suffer major defects in these regions. Alternative computational approaches will be discussed and the central issues in the associated numerical analysis of these layer-adapted algorithms will be outlined.

We present new results in the numerical analysis of singularly perturbed convection-diffusion- reaction problems that have appeared in the last five years. Mainly discussing layer-adapted meshes, we present also a survey on stabilization methods, adaptive methods, and on systems of singularly perturbed equations.

In this paper a singularly perturbed reaction-diffusion equation with a discontinuous source term is examined. A numerical method is constructed for this problem which involves an appropriate piecewise-uniform mesh. The method is shown to be uniformly convergent with respect to the singular perturbation parameter.

An exponentially-fitted method for singularly perturbed, one-dimensional parabolic equations and ordinary differential equations both of the convection-diffusion-reaction type in equally-spaced grids is presented.

Singular perturbation problems with turning points arise as mathematical models for various physical phenomena. The problem with interior turning point represent one-dimensional version of stationary convection- diffusion problems with a dominant convective term and a speed field that changes its sign in the catch basin.

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INTRODUCTION

A singular perturbation problem is a problem which depends on a parameter (or parameters) in such a uay that solutions of the problem behave nonuniformly as the parameter tends toward some limiting value of interest. Such singular perturbation problems involving differential equations arise in many areas of interest, e.g. modelling of semi-conductor devices, aero-dynamics, fluid mechanics, thin shells. We illustrate some of the nonuniformities that occur with some simple prototypes.

There are two main approaches to solving differential equations numerically:

(1) Finite Difference Methods

In one dimension, divide the interval [a,b] into N sub-intervals

$$a = x_0 < x_1 < \dots\dots\dots < x_N = b$$

Replace y and its derivatives in the differential equation by suitable (difference) approximations e.g. replace

$$y'(x_j) \text{ by } (u_{j+1} - u_j)/(x_{j+1} - x_j)$$

and then replace the coefficients of the derivatives by an appropriate approximation.

e.g.
on $[x_j, x_{j+1}]$ replace $a(x)$ by $a(x_j)$ or $a(x_{j+1})$

A system of algebraic equations is then solv/ed to generate a set of points $\{u_j\}$ as an approximation to the set $\{y(x_j)\}$

(2) Finite Element Methods

A function u(x) is generated by discretizing a weak form of the differential equation. This function approximates the solution y(x) globally.

In this note we will confine the discussion to finite difference methods.

Classical numerical methods perform badly (to say the least) when applied to singularly perturbed problems. In particular, their atrocious behaviour is most noticeable in non-self-adjoint problems.

Singularly perturbed differential equations arise in many branches of science and engineering. The solutions of such equations have boundary and interior layers. That is, there are thin layer(s) where the solution changes rapidly, while away from the layer(s) the solution behaves regularly and changes slowly. So the numerical treatment of singularly perturbed differential equations gives major computational difficulties, and in recent years, a large number of special purpose methods have been developed to provide accurate numerical solutions which cover mostly second order equations. But only a very few authors have developed numerical methods for singularly perturbed higher order differential equations [IT]. Moreover, most of them have concentrated only on the problems with smooth data. Of course, some authors have recently considered Singular Perturbation Problems (SPPs) for second order ODEs with discontinuous source term and discontinuous convection coefficient. Due to the discontinuity at one or more points in the interior domain, this gives rise to an interior layer(s) in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Therefore, these types of SPPs have to be dealt with separately and carefully. In this paper, an asymptotic numerical method for singularly perturbed react ion-diffusion type third order ODE with a discontinuous source term is developed. The classification of singularly perturbed higher order problems (reaction-diffusion/convection-diffusion) depend on how the order of the original equation is affected if one sets *e* = 0. If the order is reduced by one, we say that the problem is of convection-diffusion type, and of reaction-diffusion type if the order is reduced by two.

Systems involving several time scales often assume the prototypical form

$$\dot{x}^\epsilon = f(x^\epsilon, y^\epsilon, \epsilon)$$

$$\dot{y}^\epsilon = \frac{1}{\epsilon} g(x^\epsilon, y^\epsilon, \epsilon), \quad x^\epsilon(0) = x_0,\, y^\epsilon(0) = y_0 \qquad (1)$$

where $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$ and $0 < \epsilon \ll 1$ is a small parameter and we use the "overdot" to denote derivatives with respect to the free variable, i.e., $\dot{z}(t) = dz/dt$. If $f(\cdot,\cdot,\epsilon)$ and $g(\cdot,\cdot,\epsilon)$ are globally Lipschitz and uniformly bounded in ϵ, then $\dot{y}^\epsilon(t)$ will be of order $1/\epsilon$ faster than $\dot{x}^\epsilon(t)$. Accordingly, we call *x* the slow variables and *y* the fast variables of the system.

The analyses of singularly perturbed differential equations such as (1) often boil down to linear operator equations of the type (see, e.g., for a treatment of stochastic systems)

$$\partial\phi^\epsilon(u,t) = \mathcal{L}^\epsilon \phi^\epsilon(u,t), \quad \phi^\epsilon(u,0) = \psi(u). \qquad (2)$$

Here $\mathcal{L}^\epsilon$ is a differential operator that is defined on some Banach space subject to suitable boundary conditions and $u \in \mathbf{R}^{n+m}$ is a shorthand for *(x, y).* If we confine our attention to the aforementioned class of problems (averaging or geometric singular perturbation) the operator typically takes the form

$$\mathcal{L}^\epsilon = \mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1$$

where $\mathcal{L}_0$ and $\mathcal{L}_1$ "generate" the slow and fast dynamics, respectively. (The properties of $\mathcal{L}_i$ depend on the actual problem and will be discussed later on in the text.) Notice that $\mathcal{L}_0$ and $\mathcal{L}_1$ may still depend on ϵ, but the dominant singularity is as sketched. We seek a perturbative expansion of the solution of (2) that has the form

$$\phi^\epsilon = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots.$$

Hence (1.6) can be recast as

$$\frac{1}{\epsilon}\mathcal{L}_1\phi_0 + (\mathcal{L}_1\phi_1 + \mathcal{L}_0\phi_0 - \partial\phi_0) = \mathcal{O}(\epsilon),$$

and equating powers of ϵ yields a hierarchy of equations the first two of which are

$$\mathcal{L}_1\phi_0 \quad = \quad 0$$

$$\mathcal{L}_1\phi_1 = \partial\phi_0 - \mathcal{L}_0\phi_0 .$$

The first equation implies that the lowest-order perturbation approximation of (1) lies in the nullspace of $\mathcal{L}_1$, which typically entails a condition of the form $\phi_0 = \phi_0(x)$. Averaging or geometric singular perturbation theory now consists in finding an appropriate closure of the second equation subject to $\phi_0 \in \ker \mathcal{L}_1$. This results in an effective equation for $\phi \approx \phi_0$, namely,

$$\partial\phi(x,t) = \bar{\mathcal{L}}\phi(x,t)$$

Once the effective linear operator $\bar{\mathcal{L}}$ has been computed from $\mathcal{L}_0$ and $\mathcal{L}_1$, the result can be reinterpreted in terms of the corresponding differential equation to give $\dot{x} = \bar{f}(x)\,, \quad x(0) = x_0$

GEOMETRIC SINGULAR PERTURBATION THEORY

Consider the following system of singularly perturbed ordinary differential equation

$$\dot{x} = f(x,y)\,, \quad x(0) = x_0 \qquad (3)$$

$$\epsilon\dot{y} = g(x,y)\,, \quad y(0) = y_0$$

for $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$ aid $0 < \epsilon \ll 1$. Here and in the following we omit the parameter ϵ, i.e., we write $x = x^\epsilon$, $y = y^\epsilon$ and so on; the meaning should be always clear from the context and we indicate otherwise when not. We let $\varphi_\xi^t : \mathbf{R}^m \to \mathbf{R}^m$, $y(t) = \varphi_\xi^t(y_0)$ denote the solution of the associated system

$$\dot{y} = g(\xi,y)\,, \quad y(0) = y_0 \qquad (4)$$

and recall that the first (i.e., slow) equation in (3) can be approximately viewed as an equation of the form $\dot{x} = f(x, \varphi_x^{t/\epsilon}(y_0))\,, \quad x(0) = x_0$

Let us suppose that

$$\lim_{t\to\infty} \varphi_\xi^t(y_0) = m(\xi)$$

exists independently of the initial value $y(0) = y_0$ and uniformly in $x = \xi$, i.e., the rate of convergence is independent of the slow variable. In particular, $\lim_{\epsilon\to 0} \varphi_x^{t/\epsilon}(y_0) = m(x)$ for fixed and, by the above argument, we may replace (4.1) by the equation

$$\dot{x} = f(x, m(x))\,, \quad x(0) = x_0$$

whenever e is sufficiently small. We shall give an example. Example 4.1 Consider the linear system

$$\dot{x} = A_{11}x + A_{12}y\,, \quad x(0) = x_0$$

$$\epsilon\dot{y} = A_{21}x + A_{22}y\,, \quad y(0) = y_0 .$$

We suppose that $\sigma(A_{22}) \subset \mathbf{C}^-$, i.e., all eigenvalues of A22 lie in the open left complex half-plane. As we shall see this is equivalent to the statement that the fast subsystem is asymptotically stable. The associated system

$$\dot{y} = A_{22}(y + A_{22}^{-1}A_{21}\xi)\,, \quad y(0) = y_0$$

is easily solvable using variation of constants, viz.,

$$\varphi_\xi^t(y_0) = \exp(tA_{22})(y + A_{22}^{-1}A_{21}\xi) - A_{22}^{-1}A_{21}\xi .$$

Note that A_{22} is invertible by the stability assumption above. Furthermore

$$\varphi_\xi^t(y_0) \to -A_{22}^{-1}A_{21}\xi \quad \text{as} \quad t \to \infty$$

The vector field for a planar linear system with $A_{11} = A_{22} = -1$ and $A_{12} = A_{21} = -1/2$ is shown in Figure 1 below. The leftmost plot shows various solutions for e whereas the right figure depicts solutions (blue curves) for $\epsilon = 0.05$. It turns out that the solutions quickly converge to the null cline $\dot{y} = 0$ (fast dynamics) before converging to the asymptotically stable fixed point $(x,y) = (0,0)$ along the nullcline (slow dynamics). Note that the nullcline corresponds to the invariant subspace that is defined by the equation $A_{21}x + A_{22}y = 0$. If $A_{22} < 0$, the subspace is attractive (i.e., asymptotically stable).

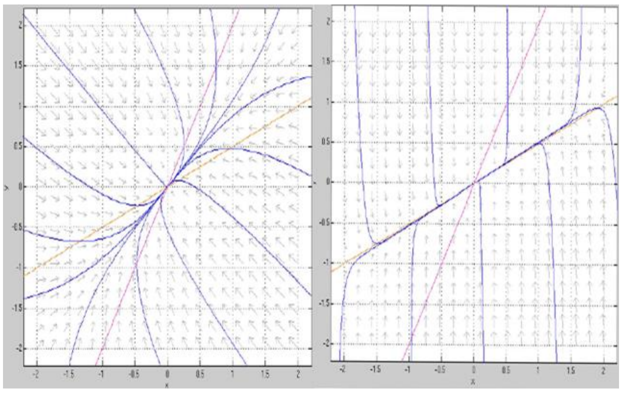

Fig. 1 Planar linear system An = A22 = —1, A12 = A21 = -1/2 for e = 1 (left panel) and e = 0.05 (right panel). The orange line shows the nullcline y = 0.

ROBUST NUMERICAL METHODS FOR SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS: A SURVEY 5. SINGULARLY PERTURBED SYSTEMS

In 2009, Linss and Stynes presented a survey on the numerical solution of singularly perturbed systems. In this Section we only comment on some recent results not contained in which sparkle that survey.

First we study systems of reaction-diffusion equations of the form

$$-E^2u'' + Au = f \quad \text{in } (0,1), u(0) = u(1) = 0, \tag{5}$$

where $E = \operatorname{diag}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_l)$ with $\varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_l$. If the matrix *A* satisfies certain conditions, the asymptotic behavior of such a system is well understood. Assume that *A* has positive diagonal entries, moreover the matrix Γ defined by

$$\gamma_{ii} = 1, \quad \gamma_{ij} = -\left\|\frac{a_{ij}}{a_{ii}}\right\|_\infty \quad \text{for } i \ne j \tag{6}$$

satisfies $\Gamma^{-1} \ge 0$. Then, in the existence of a solution decomposition is proved. Other authors assume that *A* is an M-matrix or that *A* is point-wise positive definite. See for establishing a connection between positive definiteness and the property $\Gamma^{-1} \ge 0$. In a full asymptotic expansion is derived for positive definite *A* in the case of two equations, including information on analytic regularity.

Systems of convection-diffusion problems are more delicate to handle. Consider first weakly coupled systems of the form

$$Lu := -Eu'' - \operatorname{diag}(b)u' + Au = f, \quad u(0) = u(1) = 0, \tag{7}$$

assuming $|b_i| \ge \beta_i > 0$ and $\tilde{\Gamma}^{-1} \ge 0$ with

$$\tilde{\gamma}_{ii} = 1, \quad \tilde{\gamma}_{ij} = -\min\left(\left\|\frac{a_{ij}}{a_{ii}}\right\|_\infty, \left\|\frac{a_{ij}}{b_i}\right\|_\infty\right) \quad \text{for } i \ne j. \tag{8}$$

Then it was shown in for $\nu = 0, 1$

$$\left|u_k^{(\nu)}(x)\right| \le C\begin{cases} 1 + \varepsilon_k^{-\nu} e^{-\beta_k(1-x)/\varepsilon} & \text{if } b_k < 0, \\ 1 + \varepsilon_k^{-\nu} e^{-\beta_k x/\varepsilon} & \text{if } b_k > 0. \end{cases} \tag{9}$$

When only first order derivatives are considered, there is no strong interaction between the layers of different components unlike the reaction-diffusion case.

But, consider for example a set of two equations with $b_1 > 0$ and $b_2 < 0$ for $\varepsilon_1 = \varepsilon_2$.

Then the layer at *x* = 1 in the first component generates a weak layer at *x* = 1 in the second component; the situation at* = 0 is analogous. Under certain conditions , one can prove the existence of the following solution decomposition for $\nu \le 2$:

$$u_1 = S_1 + E_{10} + E_{11}, \tag{10}$$

$$u_2 = S_2 + E_{20} + E_{21}$$

$$\left\|S_1^{(\nu)}\right\|_0, \left\|S_2^{(\nu)}\right\|_0 \le C,$$
$$\left|E_{10}^{(\nu)}(x)\right| \le C\varepsilon^{1-\nu}e^{-\alpha(x/\varepsilon)}, \quad \left|E_{11}^{(\nu)}(x)\right| \le C\varepsilon^{-\nu}e^{-\alpha((1-x)/\varepsilon)},$$
$$\left|E_{20}^{(\nu)}(x)\right| \le C\varepsilon^{-\nu}e^{-\alpha(x/\varepsilon)}, \quad \left|E_{21}^{(\nu)}(x)\right| \le C\varepsilon^{1-\nu}e^{-\alpha((1-x)/\varepsilon)}. \qquad 11)$$

Here α is some positive parameter. This observation is important for control problems governed by convection-diffusion equations:

$$\min_{y,q} J(y,q) := \min_{y,q}\left(\frac{1}{2}\|y - y_0\|_0^2 + \frac{\lambda}{2}\|q\|_0^2\right) \tag{12}$$

subject to

$$Ly := -\varepsilon y'' + by' + cy = f + q \quad \text{in } (0,1),$$
$$y(0) = y(1) = 0. \tag{13}$$

For strongly coupled systems of convection-diffusion equations full layer-interaction takes place. Consider the system of two equations

$$Lu := -\varepsilon u'' - Bu' + Au = f, \quad u(0) = u(1) = 0 \tag{14}$$

assuming

(Vi) B is symmetric.

(V_2) $A + 1/2B'$ is positive semidefinite.

(V_3) The eigenvalues of B satisfy $|\lambda_{1,2}| > \alpha > 0$ for all x.

If both eigenvalues of B are positive, both solution components do have overlapping layers at $x = 0$, the reduced solution solves an initial value problem. But if the eigenvalues do have a different sign, both solution components do have layers at $x = 0$ *and* $x = 1$; we have full layer interaction. It is remarkable that the reduced solution, in general, does not satisfy any of the given boundary conditions.

Even more complicated are strongly coupled systems with several small parameters of the form

$$Lu := -Eu'' - Bu' + Au = f, \quad u(0) = u(1) = 0. \tag{15}$$

Some a priori estimates are to find in, information on the layer structure in.

A REVIEW ON SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS WITH TURNING POINTS AND INTERIOR LAYERS

Many phenomena in biology, chemistry, engineering, physics, etc., can be described by boundary value problems associated with various type of differential equations or systems. Whenever a mathematical model is associated with a phenomenon, the researchers generally try to capture what is essential, retaining the important quantities and omitting the negligible ones which involve small parameters. The model that would be obtained by maintaining the small parameters is called the perturbed model, whereas the simplified model (the one that does not include the small parameters) is called the unperturbed (or reduced) model. For study purpose the perturbed model can be replaced by its unperturbed counterpart but what matters is that its solution must be "close enough" to the solution of the corresponding perturbed model. This fact holds good in case of regular perturbation but, in the case of singular perturbation it is very unlikely to hold.

These singular perturbation problems with or without turning point(s) commonly occur in many branches of applied mathematics, e.g., as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics. In these kind of problems perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid change. These narrow regions frequently adjoin the boundaries or some interior point of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Therefore, these kind of problems exhibit boundary and/or interior layers, i.e., there are thin regions where the solution changes rapidly.

Kadalbajoo and Reddy gave survey of various asymptotic and numerical methods developed from 1908 — 1986 for the determination of approximate solution of singular perturbation problems of various kinds. Kadalbajoo and Patidar extended the work done by Kadalbajoo and Reddy and surveyed the work done by various researchers in the area of singular perturbation from 1984 — 2000. In this work they considered one dimensional problem only and discussed the work done on linear, non-linear, semilinear and quasilinear problems. In Kadalbajoo and Patidar covered the survey of singularly perturbed partial differential equations and surveyed the work done in this area from 1980 — 2000. Kadalbajoo and Vikas

in continuation with the work done by the first author gave brief survey on the computational techniques for different classes of singular perturbation problems considered by various researchers from 2000 — 2009. In this way one can see that this area has developed so much in the past century that it is not possible to give whole of the survey in a single paper. In particular, singularly perturbed differential equations with turning point form an important class of problems which are very challenging and even today there is a lot to be explored in this area. Also, problems where discontinuity in the data results into interior layers in the solution of the problem commonly occur during modeling of physical processes. So. we thought that it will be good to write a survey which exclusively gives details about the work done in these two areas till date. In this work we restrict ourselves to the study of singularly perturbed turning point problems whose solution possesses either boundary or interior layers in the turning point region. We also consider some works in which non-smooth solutions occur inside the domain. We tried to give as much information as we can but if we miss some important names and papers that is purely unintentional.

Singular perturbation problems with turning points arise as mathematical models for various physical phenomena. Among these, the problem with interior turning points represent one-dimensional version of stationary convection-diffusion problems with a dominant convective term and a speed field that changes its sign in the catch basin. Boundary turning point problems, on the other hand, arise in geophysics and in the modeling of thermal boundary layers in laminar flow. The problem from models heat flow and mass transport near an oceanic rise. It is a single boundary turning point problem because of the assumption that the velocity distribution is linear. If one allows for higher order of velocity distribution, then it becomes multiple boundary turning point problems.

A typical linear turning point problem in one dimension is given by

$$\varepsilon y'' + a(x)y' + b(x)y = 0, \quad x \in [x_1, x_2], \quad x_1 < 0, \ x_2 > 0,$$

where $0 < \varepsilon \leq 1, \ a(x)$, $b(x)$ are sufficiently smooth. This problem has acquired great amount of interest of mathematicians as well as physicist due to the fact that the solution exhibit some interesting behavior such as boundary layer, interior layer and resonance phenomena. When $a(x)$ does not change sign in the entire interval $[x_1, x_2]$, the solution is characterized by a boundary layer near one endpoint as $\varepsilon \to 0$. When $a(x)$ has a simple zero, say at $x_0 = 0$ in $[x_1, x_2]$, the point x_0 is the so-called turning point and the problem is classified as turning point problem? In this situation the solution behavior depend upon the properties of the coefficient functions $a'(x)$ and $b(x)$ at the turning point xo = 0. Indeed, if it is assumed that, for α, β constants, $a(x) \sim \alpha x$ and $b(x) \sim \beta$ as $x \to 0$ the following cases arise:

(i) If $\alpha > 0, \ \beta/\alpha \neq 1, \ 2, \ 3, \dots$, an internal layer occurs near the turning point $x_0 = 0$

(ii) If $\alpha < 0, \ \beta/\alpha \neq 0, \ -1, \ -2, \dots$, then there are two boundary layers appearing at the two endpoints of the interval.

(iii) If $\alpha < 0$ and $\beta/\alpha = 0$, $-1, \ -2, \dots$, or if $\alpha > 0$, $\beta/\alpha = 1, \ 2, \ 3, \dots$, the solution exhibit a very interesting phenomenon named as Ackerberg-O'Malley's resonance phenomenon .

Another situation where interior layer arise would be the case of singularly perturbed convection- diffusion-reaction problems based on non-smooth data. If one or more coefficients such as the convection term, reaction term, source term or the boundary conditions are discontinuous, the solution of such type of problems exhibit strong or weak interior layers depending on the magnitude of the singular perturbation parameter and the nature of the coefficients.

There are two principle approaches for solving singular perturbation problems: Numerical approach and Asymptotic approach. Asymptotic approach helps us to gain insight into the qualitative behavior of the problem and give only semi quantitative information, whereas numerical approach provides quantitative information about the particular member of the family of solutions. Numerical methods can be applied to a broader class of problems and are intended to minimize demands upon the problem solver. Asymptotic methods require the problem solver to have some knowledge of the behavior of the expected solution and treat comparatively restricted class of problems. There are few researchers who combined both the above approaches to yield better results. Systematic study of turning point problems started in late 60s. Since then this subject has matured quite a lot and many textbooks have appeared in the area of singular perturbations which discussed turning point problems and dealt with either asymptotic approach or numerical approach or considered both of them, see, e.g., - .

SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS SOURCE TERMS

In this paper a singularly perturbed reaction - diffusion equation in one dimension with a discontinuous source term is considered on the unit interval $\Omega = (0, 1)$. A single discontinuity in the source term is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation $\Omega^- = (0, d)$ and $\Omega^+ = (d, 1)$ and to denote the jump at d in any function with $[\omega](d) = \omega(d+) - \omega(d-)$. The corresponding self-adjoint two point boundary value problem is

$$(P_\varepsilon) \begin{cases} \text{Find} \quad u_\varepsilon \in C^1(\bar{\Omega}) \cap C^2(\Omega^- \cup \Omega^+) \quad \text{such that} \\ -\varepsilon u_\varepsilon'' + a(x)u_\varepsilon = f \quad \text{for all} \quad x \in \Omega^- \cup \Omega^+ \\ u_\varepsilon(0) = u_0, \quad u_\varepsilon(1) = u_1 \\ f(d-) \neq f(d+), \quad a(x) \geq \alpha > 0 \end{cases}$$

where a and f are sufficiently smooth on $\bar{\Omega} \setminus \{d\}$. Because f is discontinuous at cl the solution u_ε of (P_ε) does not necessarily have a continuous second order derivative at the point *d*. Thus $u_\varepsilon \notin C^2(\Omega)$, but the first derivative of the solution exists and is continuous.

Theorem 1 *the problem* (P_ε) *has a solution* $u_\varepsilon \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ Proof: The proof is by construction. Let 2/1,2/2 be particular solutions of the differential equations

$$-\varepsilon y_1'' + a(x)y_1 = f, \ x \in \Omega^-,$$

and

$$-\varepsilon y_2'' + a(x)y_2 = f,\ x \in \Omega^+.$$

Consider the function

$$y(x) = \begin{cases} y_1(x) + (u_\varepsilon(0) - y_1(0))\phi_1(x) + A\phi_2(x), & x \in \Omega^- \\ y_2(x) + B\phi_1(x) + (u_\varepsilon(1) - y_2(1))\phi_2(x), & x \in \Omega^+ \end{cases}$$

where $\phi_1(x), \phi_2(x)$ are the solutions of the boundary value problems

$$-\varepsilon\phi_1'' + a(x)\phi_1 = 0,\ x \in \Omega,\quad \phi_1(0) = 1,\ \phi_1(1) = 0$$
$$-\varepsilon\phi_2'' + a(x)\phi_2 = 0,\ x \in \Omega,\quad \phi_2(0) = 0,\ \phi_2(1) = 1$$

and *A. D* are constants to be chosen so that $y \in C^1(\Omega)$. Note that on the open interval $(0, 1)$, $0 < \phi_i < 1$, $i = 1, 2$.

Thus ϕ_1, ϕ_2 cannot have an internal maximum or minimum and hence

$$\phi_1' < 0,\quad \phi_2' > 0 \quad x \in (0, 1).$$

We wish to choose the constants *A. D* so that $y \in C^1(\Omega)$. That is we impose $y(d^-) = y(d^+)$, and $y'(d^-) = y'(d^+)$.

For the constants *A, D* to exist we require $\begin{vmatrix} \phi_2(d) & -\phi_1(d) \\ \phi_2'(d) & -\phi_1'(d) \end{vmatrix} \neq 0$

This follows from observing that $\phi_2'(d)\phi_1(d) - \phi_2(d)\phi_1'(d) > 0.$

CONCLUSION

With the advent of robust computational solvers, we can now solve very complex systems of differential equations which, perhaps, cannot even be analyzed using the asymptotic methods. Since singular perturbation problems are parameter dependent, the use of numerical methods developed for solving regular problems leads to errors in the solution that depend on the value of the parameter ε. Errors of the numerical solution depend on the distribution of mesh points and become small only when the effective mesh-size in the layer is much less than the value of the parameter ε. Such numerical methods turn out to be inapplicable for singular perturbation problems. Due to this, there arises interest in the development of parameter-uniform.

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