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A Study on Structures and Invariants of Rings

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Abstract – In mathematics, a ring is an algebraic structure consisting of a set together with two binary operations usually called addition and multiplication, where the set is an abelian group under addition (called the additive group of the ring) and a monoid under multiplication such that multiplication distributes over addition.

In other words the ring axioms require that addition is commutative, addition and multiplication are associative, multiplication distributes over addition, each element in the set has an additive inverse, and there exists an additive identity. One of the most common examples of a ring is the set of integers endowed with its natural operations of addition and multiplication. Certain variations of the definition of a ring are sometimes employed, and these are outlined later in the article.

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INTRODUCTION

The branch of mathematics that studies rings is known as ring theory. Ring theorists study properties common to both familiar mathematical structures such as integers and polynomials, and to the many less well-known mathematical structures that also satisfy the axioms of ring theory. The ubiquity of rings makes them a central organizing principle of contemporary mathematics. Ring theory may be used to understand fundamental physical laws, such as those underlying special relativity and symmetry phenomena in molecular chemistry. The concept of a ring first arose from attempts to prove Fermat's last theorem, starting with Richard Dedekind in the 1880s. After contributions from other fields, mainly number theory, the ring notion was generalized and firmly established during the 1920s by Emmy Noether and Wolfgang Krull. Modern ring theory - a very active mathematical discipline - studies rings in their own right. To explore rings, mathematicians have devised various notions to break rings into smaller, better-understandable pieces, such as ideals, quotient rings and simple rings. In addition to these abstract properties, ring theorists also make various distinctions between the theory of commutative rings and non-commutative rings- the former belonging to algebraic number theory and algebraic geometry. A particularly rich theory has been developed for a certain special class of commutative rings, known as fields, which lies within the realm of field theory. Likewise, the corresponding theory for non-commutative rings, that of non-commutative division rings, constitutes an active research interest for non-commutative ring theorists. Since the discovery of a mysterious connection between non-commutative ring theory and geometry during the 1980s by Alain Connes, non-commutative geometry has become a particularly active discipline in ring theory.

Category theoretical description Every ring can be thought of as a monoid in Ab, the category of abelian groups (thought of as a monoidal category under the tensor product of -modules). The monoid action of a ring R on an abelian group is simply an R-module. Essentially, an R-module is a generalization of the notion of a vector space - where rather than a vector space over a field, one has a "vector space over a ring". Let (A, +) be an abelian group and let End(A) be its endomorphism ring (see above). Note that, essentially, End(A) is the set of all morphisms of A, where if f is in End(A), and g is in End(A), the following rules may be used to compute f + g and f · g: • (f + g)(x) = f(x) + g(x) • (f · g)(x) = f(g(x)) where + as in f(x) + g(x) is addition in A, and function composition is denoted from right to left. Therefore, associated to any abelian group, is a ring. Conversely, given any ring, (R, +, ·), (R, +) is an abelian group. Furthermore, for every r in R, right (or left) multiplication by r gives rise to a morphism of (R, +), by right (or left) distributivity. Let A = (R, +). Consider those endomorphisms of A, that "factor through" right (or left) multiplication of R. In other words, let EndR (A) be the set of all morphisms m of A, having the property that m(r · x) = r · m(x). It was seen that every r in R gives rise to a morphism of A - right multiplication by r. It is in fact true that this association of any element of R, to a morphism of A, as a function from R to End R (A), is an isomorphism of rings. In this sense, therefore, any ring can be viewed as the endomorphism ring of some abelian X-group (by X-group, it is meant a group with X being its set of operators).In essence, the most general form of a ring, is the endomorphism group of some abelian X-group.

DIMENSION OF A COMMUTATIVE RING

The Krull dimension of a commutative ring *R* is the supremum of the lengths *n* of all the increasing chains of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$. For example, the polynomial ring $k[t_1, \cdots, t_n]$ over a field *k* has dimension *n*. The fundamental theorem in the dimension theory states the following numbers coincide for a noetherian local ring $(R, \mathfrak{m})$:

- The Krull dimension of *R*.
- The minimum number of the generators of the $\mathfrak{m}$-primary ideals.
- The dimension of the graded ring $\operatorname{gr}_{\mathfrak{m}}(R) = \oplus_{k \geq 0} \mathfrak{m}^k / \mathfrak{m}^{k+1}$ (equivalently, one plus the degree of its Hilbert polynomial).

A commutative ring *R* is said to be catenary if any pair of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ can be extended to a chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}'$ of same finite length such that there is no prime ideal that is strictly contained in two consecutive terms. Practically all noetherian rings that appear in application are catenary. If $(R, \mathfrak{m})$ is a catenary local integral domain, then, by definition,

$$\dim R = \operatorname{ht} \mathfrak{p} + \dim R/\mathfrak{p}$$

where $\operatorname{ht} \mathfrak{p} = \dim R_{\mathfrak{p}}$ is the height of $\mathfrak{p}$. It is a deep theorem of Ratliff that the converse is also true.

If *R* is an integral domain that is a finitely generated *k*-algebra, then its dimension is the transcendence degree of its field of fractions over *k*. If *S* is an integral extension of a commutative ring *R*, then *S* and *R* have the same dimension.

Closely related concepts are those of depth and global dimension. In general, if *R* is a noetherian local ring, then the depth of *R* is less than or equal to the dimension of *R*. When the equality holds, *R* is called a Cohen–Macaulay ring. A regular local ring is an example of a Cohen–Macaulay ring. It is a theorem of Serre that *R* is a regular local ring if and only if it has finite global dimension and in that case the global dimension is the Krull dimension of *R*. The significance of this is that a global dimension is a homological notion.

MORITA EQUIVALENCE

Two rings *R*, *S* are said to be Morita equivalent if the category of left modules over *R* is equivalent to the category of left modules over *S*. In fact, two commutative rings which are Morita equivalent must be isomorphic, so the notion does not add anything new to the category of commutative rings. However, commutative rings can be Morita equivalent to non-commutative rings, so Morita equivalence is coarser than isomorphism. Morita equivalence is especially important in algebraic topology and functional analysis.

Finitely generated projective module over a ring and Picard group

Let *R* be a commutative ring and $\mathbf{P}(R)$ the set of isomorphism classes of finitely generated projective modules over *R*; let also $\mathbf{P}_n(R)$ subsets consisting of those with constant rank *n*. (The rank of a module *M* is the continuous function $\operatorname{Spec} R \to \mathbb{Z}, \mathfrak{p} \mapsto \dim M \otimes_R k(\mathfrak{p})$ $\mathbf{P}_1(R)$ is usually denoted by Pic(*R*). It is an abelian group called the Picard group of *R*. If *R* is an integral domain with the field of fractions *F* of *R*, then there is an exact sequence of groups:

$$1 \to R^* \to F^* \overset{f \mapsto fR}{\to} \operatorname{Cart}(R) \to \operatorname{Pic}(R) \to 1$$

where $\operatorname{Cart}(R)$ is the set of fractional ideals of *R*. If *R* is a regular domain (i.e., regular at any prime ideal), then Pic(R) is precisely the divisor class group of *R*.

For example, if *R* is a principal ideal domain, then Pic(*R*) vanishes. In algebraic number theory, *R* will be taken to be the ring of integers, which is Dedekind and thus regular. It follows that Pic(*R*) is a finite group (finiteness of class number) that measures the deviation of the ring of integers from being a PID.

One can also consider the group completion of $\mathbf{P}(R)$; this results in a commutative ring K_0(R). Note that K_0(R) = K_0(S) if two commutative rings *R*, *S* are Morita equivalent.

STRUCTURE OF NON-COMMUTATIVE RINGS

The structure of a non-commutative ring is more complicated than that of a commutative ring. For example, there exist simple rings, containing no non-trivial proper (two-sided) ideals, which contain non-trivial proper left or right ideals. Various invariants exist for commutative rings, whereas invariants of non-commutative rings are difficult to find. As an example, the nil-radical of a ring, the set of all nilpotent elements, need not be an ideal unless the ring is commutative. Specifically, the set of all nilpotent elements in the ring of all *n* x *n* matrices over a division ring never forms an ideal, irrespective of the division ring chosen. There are, however, analogues of the nil radical defined for non-commutative rings, that coincide with the nil-radical when commutativity is assumed.

The concept of the Jacobson radical of a ring; that is, the intersection of all right/left annihilators of simple right/left modules over a ring, is one example. The fact that the Jacobson radical can be viewed as the intersection of all maximal right/left ideals in the ring, shows how the internal structure of the ring is reflected by its modules. It is also a fact that the intersection of all maximal right ideals in a ring is the same as the intersection of all maximal left ideals in the ring, in the context of all rings; whether commutative or non-commutative.

Non-commutative rings serve as an active area of research due to their ubiquity in mathematics. For instance, the ring of *n*-by-*n* matrices over a field is non-commutative despite its natural occurrence in geometry, physics and many parts of mathematics. More generally, endomorphism rings of abelian groups are rarely commutative, the simplest example being the endomorphism ring of the Klein four-group.

APPLICATIONS

The ring of integers of a number field

The coordinate ring of an algebraic variety

If *X* is an affine algebraic variety, then the set of all regular functions on *X* forms a ring called the coordinate ring of *X*. For a projective variety, there is an analogous ring called the homogeneous coordinate ring. Those rings are essentially the same things as varieties: they correspond in essentially a unique way. This may be seen via either Hilbert's Null stellensatz or scheme-theoretic constructions (i.e., Spec and Proj).

Ring of invariants

A basic (and perhaps the most fundamental) question in the classical invariant theory is to find and study polynomials in the polynomial ring $k[V]$ that are invariant under the action of a finite group (or more generally reductive) *G* on *V*. The main example is the ring of symmetric polynomials: symmetric polynomials are polynomials that are invariant under permutation of variable. The fundamental theorem of symmetric polynomials states that this ring is $R[\sigma_1, \dots, \sigma_n]$ where σ_i are elementary symmetric polynomials

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