Journal of Advances in Science and Technology

Vol. VIII, Issue No. XV, November-2014, ISSN 2230-9659

REVIEW ARTICLE

A STUDY ON RANK EQUALITIES FOR IDEMPOTENT MATRICES

Journal of Advances in Science and Technology
Vol. VIII, Issue No. XV, November-2014, ISSN 2230-9659

A Study on Rank Equalities for Idempotent Matrices

Subhash Singh

Research Scholar of Swami Vivekanand University, MP

---------------------------♦-----------------------------

INTRODUCTION

A square matrix A over the complex field C is said to be idempotent if $A^2=A$; said to be an orthogonal projector if it is both idempotent and Hermitian, i.e., $A^2=A=A^*$, where A^* denotes the conjugate transpose of A. From the similarity theory of matrices, any idempotent matrix A can be decomposed as $A = U\,\mathrm{diag}(I_k, 0)U^{-1}$, where I_k is the identity matrix of order k and k is the rank of A; any orthogonal projector A can be decomposed as $A = U\,\mathrm{diag}(I_k, 0)U^*$, where $U^{-1}=U^*$. Idempotent matrices and orthogonal projectors appear almost everywhere, and have been the objects of many studies in matrix theory and its applications.

Idempotent matrices and orthogonal projectors also have close links with generalized inverses of matrices. For instance, both AA^- and A^-A are idempotent for any generalized inverse A^- of A ; both $AA^\dagger$ and $A^\dagger A$are orthogonal projectors for the Moore–Penrose inverse $A^\dagger$ of A . Many other types of matrices can be converted into idempotent matrices through some elementary operations. For instance, if $A^2=-A$, then $(-A)^2=-A$, i.e., -A is idempotent; if $A^2=I_m$, then $(I_m\pm A)/2$ are idempotent; if $A^2=-I_m$, then $(I_m\pm iA)/2$are idempotent. In general, any matrix A satisfying a quadratic equation $A^2+aA+bI_m=0$ can be written as $[A-(a/2)I_m]^2=(a^2/4-b)I_m$. If $a^2/4-b\neq 0$, then one can also write out an idempotent matrix from this equality.

In the investigation of idempotent matrices and their applications, one often encounters various matrix expressions consisting of idempotent matrices. For example, PQ , P±Q, $\lambda_1P+\lambda_2Q$, PA-AQ, I_m-PQ,PQ±QP, $(PQ)^2$-PQ, $AA^\dagger\pm A^\dagger A$, $AA^-\pm B^-B$, where P and Q are two idempotent matrices. On the other hand, one also considers matrix decompositions associated with idempotent matrices, like $A=P_1\pm P_2$, $A=P_1P_2\pm P_2P_1$, $A=P_1\cdots P_k$; where $P_1,P_2,\ldots,P_k$ are idempotents. In such situations, it is of interest to give some basic properties of these matrix expressions, as well as relationships among these matrix expressions.

When investigating these problems, we have noticed that the rank of matrix is a very rich technique for dealing with matrix expressions consisting of idempotent matrices. The rank of a matrix is invariant with respect to some basic operations for this matrix, such as, elementary matrix operations and similarity transformations. A well-known fact about matrix rank is: two matrices A and B of the same size are equivalent, i.e., there are two invertible matrices U and V such that UAV=B, if and only if r(A)=r(B). The simplest method for determining the linear independency of columns or rows of a matrix, as well as the rank of the matrix is to reduce the matrix to column or row echelon forms by elementary matrix operations.

Theoretically, for any matrix expression consisting of idempotent matrices, one can establish some rank equalities associated with this expression. From these rank equalities, one can derive some basic properties on this expression.

Rank formulas can be established through various block matrices and elementary block matrix operations. Some well-known results are given below:

$$r\begin{bmatrix} I_m & A \\ B & I_n \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & I_n - BA \end{bmatrix} = r\begin{bmatrix} I_m - AB & 0 \\ 0 & I_n \end{bmatrix},$$

$$r\begin{bmatrix} I_m & I_m - AB \\ B & 0 \end{bmatrix} = r\begin{bmatrix} I_m & 0 \\ 0 & B - BAB \end{bmatrix} = r\begin{bmatrix} 0 & I_m - AB \\ B & 0 \end{bmatrix},$$

$$r\begin{bmatrix} A & AB \\ BA & B \end{bmatrix} = r\begin{bmatrix} A & 0 \\ 0 & B - BAB \end{bmatrix} = r\begin{bmatrix} A - ABA & 0 \\ 0 & B \end{bmatrix}.$$

Rank equalities for idempotent matrices

Suppose P and Q are a pair of idempotents (including idempotent matrices over an arbitrary field F, idempotent operators on Banach and Hilbert spaces, idempotents in unital rings). The two fundamental operations for P and Q are given by P±Q. In many situations, it is necessary to know various properties onP±Q, for example, the non-singularity, idempotency, tripotency and nilpotency of P±Q. Various results onP±Q and their properties can be found in the literature. Recall that a square matrix A of order m is nonsingular if and only if r(A)=m. If some rank equalities and inequalities for P±Q can be established, one can derive a variety of properties for P±Q from these rank equalities. A group of valuable rank equalities for a sum of two idempotent matrices are presented, but their proofs are omitted there. Here we restate these rank equalities and give their proofs.

Theorem

Let P and Q be a pair of complex idempotent matrices of order m. Then the sum P+Qsatisfies the following rank equalities

$$r(P+Q)=r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix}-r(Q)=r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix}-r(P),$$

r(P+Q)=r[P-PQ,Q]=r[Q-QP,P],

$$r(P+Q)=r\begin{bmatrix} P-QP \\ Q \end{bmatrix}=r\begin{bmatrix} Q-PQ \\ P \end{bmatrix},$$

r(P+Q)=r(P-PQ-QP+QPQ)+r(Q),

r(P+Q)=r(Q-PQ-QP+PQP)+r(P).

Proof

Recall that the rank of a matrix is an important invariant quantity under elementary matrix operations for this matrix, that is, these operations do not change the rank of the matrix. Thus, we first find by elementary block matrix operations the following trivial result:

$$r\begin{bmatrix} P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix}=r\begin{bmatrix} P & 0 & 0 \\ 0 & Q & 0 \\ 0 & 0 & -P-Q \end{bmatrix}=r(P)+r(Q)+r(P+Q).$$

On the other hand, note $P^2=P$ and $Q^2=Q$. By elementary block matrix operations, we also obtain another nontrivial rank equality

$$r\begin{bmatrix} P & 0 & P \\ 0 & Q & Q \\ P & Q & 0 \end{bmatrix}=r\begin{bmatrix} P & 0 & P \\ -QP & 0 & Q \\ P & Q & 0 \end{bmatrix}=r\begin{bmatrix} 2P & 0 & P \\ 0 & 0 & Q \\ P & Q & 0 \end{bmatrix}$$

$$=r\begin{bmatrix} 2P & 0 & 0 \\ 0 & 0 & Q \\ 0 & Q & \frac{1}{2}P \end{bmatrix}$$

$$=r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix}+r(P).$$

Theorem

Let P and Q be a pair of idempotent matrices of order m, and let $_{a1}$and $_{a2}$be two nonzero scalars such that $_{a1}+_{a2}\neq 0$.Then $r(_{a1}P+_{a2}Q)=r(P+Q)$, that is, the rank of $_{a1}P+_{a2}Q$is invariant with respect to $_{a1}\neq 0,_{a2}\neq 0$and $_{a1}+_{a2}\neq 0$.

Equality was proposed by Tian as a problem and solved by Bataille and other seven solvers. Equality can be derived from a result in Tian and Styan that for any $P^2=\lambda P$ and $Q^2=\lambda Q$ with$\lambda P\neq 0$ and $\mu\neq 0$,

$$r(\mu P+\lambda Q)=r\begin{bmatrix} P & Q \\ Q & 0 \end{bmatrix}-r(Q)=r\begin{bmatrix} Q & P \\ P & 0 \end{bmatrix}-r(P).$$

Theorem

Let P and Q be a pair of idempotent matrices of order m. Then equation

$$r(P+Q)=r(P)+r(Q)-m+r\left(\begin{bmatrix} I_m-P \\ I_m-Q \end{bmatrix}[I_m-P, I_m-Q]\right).$$

Proof

By elementary block matrix operations

$$r\begin{bmatrix} P & 0 & I_m \\ 0 & Q & I_m \\ I_m & I_m & 0 \end{bmatrix}=r\begin{bmatrix} 0 & 0 & I_m \\ 0 & P+Q & 0 \\ I_m & 0 & 0 \end{bmatrix}=2m+r(P+Q).$$

Also find by elementary block matrix operations

$$r\begin{bmatrix} P & 0 & I_m \\ 0 & Q & I_m \\ I_m & I_m & 0 \end{bmatrix}=r\begin{bmatrix} P & 0 & I_m-P \\ 0 & Q & I_m-Q \\ I_m-P & I_m-Q & -2I_m \end{bmatrix}$$

$$=r(P)+r(Q)+r\begin{bmatrix} 0 & 0 & I_m-P \\ 0 & 0 & I_m-Q \\ I_m-P & I_m-Q & -2I_m \end{bmatrix}$$

$$=r(P)+r(Q)+r\begin{bmatrix} 0 & 0 & I_m-P \\ 0 & 0 & I_m-Q \\ I_m-P & I_m-Q & I_m \end{bmatrix}$$

$$=r(P)+r(Q)+r\begin{bmatrix} I_m-P & (I_m-P)(I_m-Q) & 0 \\ (I_m-Q)(I_m-P) & I_m-Q & 0 \\ 0 & 0 & I_m \end{bmatrix}$$

$$=r(P)+r(Q)+m+r\left(\begin{bmatrix} I_m-P \\ I_m-Q \end{bmatrix}[I_m-P, I_m-Q]\right).$$

REFERENCES

- Baksalary. J.K, Pukelsheim. F and Styan. G.P.H, “Some properties of matrix partial orderings”, Lin. Alg. Appl., vol.119, (2009) : 7 85.

- Baksalary. O.M, “Idem potency of linear combinations of three idempotent matrices, two of which are disjoint”, Lin. Alg. Appl., 388, (2004) : 67 78.

- Ballantine. C.S, “Products of idempotent matrices”, Lin. Alg. Appl., 19, (2008) : 81 86.

- Ben Israel. A and Greville.T.N.E, "Generalized inverses, Theory and applications", Wiley and sons, New York, 2004.
- Ben Israel. A and Greville.T. N. E, "Generalized inverses, Theory and applications" Springerverlag, New York, 2003.