Journal of Advances in Science and Technology

Vol. VIII, Issue No. XV, November-2014, ISSN 2230-9659

A STUDY ON TIME DEPENDENT COORDINATE TRANSFORMATION AND ITS APPLICATIONS

AN INTERNATIONALLY INDEXED PEER REVIEWED & REFEREED JOURNAL

www.ignited.in

Journal of Advances in Science and Technology
Vol. VIII, Issue No. XV, November-2014, ISSN 2230-9659

A Study on Time Dependent Coordinate Transformation and Its Applications

Rinky

Research Scholar of Singhania University, Jhunjhunu, Rajasthan

Abstract – This research is primarily concerned with the use of Time Dependent Coordinate Transformation methods for the approximate solution of nonlinear partial differential equations of parabolic type. Such methods have become a popular means for the solution of problems which may contain sharp features that are hard to approximate, whilst efficiently managing computational overheads.

---------------------------♦-----------------------------

INTRODUCTION

Time Dependent Coordinate Transformation method is developed initially powered by conservation of mass and later by equi-distribution principles. The unusual feature of the method is that the resulting system is solved entirely through the grid coordinates with the underlying partial differential equation solution being constructed algebraically a posteriori from the mesh. Numerical results are compared with an analytical solution to the porous media equation which drives the development of the method. Self-similar theory is also used to verify our approximate solutions.

The method of Variation of Parameters is a much more general method that can be used in many more cases. However, there are two disadvantages to the method. First, the complementary solution is absolutely required to do the problem. This is in contrast to the method of undetermined coefficients where it was advisable to have the complementary solution on hand, but was not required. Second, as we will see, in order to complete the method we will be doing a couple of integrals and there is no guarantee that we will be able to do the integrals. So, while it will always be possible to write down a formula to get the particular solution, we may not be able to actually find it if the integrals are too difficult or if we are unable to find the complementary solution.

In one dimension the method is highly successful when applied to nonlinear diffusion problems incorporating moving or fixed boundaries and problems with blow up.

However in higher dimensions we encounter oscillatory solutions emanating from an inability to manipulate our solution technique into square invertible systems which satisfy all of the conditions required for this style of mesh movement.

The use of adapted meshes in the numerical solution of partial differential equations (PDEs) has become a popular technique for improving existing approximation Schemes. In problems in which features with large solution variations are common such as steep fronts and sharp variations, the choice of a non-uniform mesh can not only retain the accuracy but also improve the efficiency of an existing method by concentrating mesh points within regions of interest. This thesis is primarily concerned with the use of such methods for the solution of nonlinear parabolic PDEs of the form

$$u_t = (D(u)u_x)_x + Q(u).$$

In particular, we shall be considering problems involving nonlinear diffusion and solution blow-up. The advantages of such an approach go hand in hand. Firstly, since such areas of interest in the mesh inevitably involve large variations in the solution for any numerical scheme a smaller spatial resolution in the mesh is essential for a reliable approximation and accurate representation. However to enforce this requirement over the entire grid will be an expensive process especially in higher dimensions. It becomes obvious then that for efficiency it is desirable to concentrate nodes and hence computational effort in those parts of the grid that require most attention.

A successful approach will then ensure suitable mesh resolution whilst retaining computational efficiency. In general there are three classifications of grid adaption. The first h refinement adds extra nodes to an existing mesh to improve local grid resolution. A second technique, P refinement employs higher order numerical schemes to improve local accuracy as well as to approximate troublesome derivatives. The third approach is r refinement which maintains the existing number of nodes globally but relocates them strategically and more importantly efficiently over the domain. It is this latter idea that we shall be concerned with in the course of this research work.

In view of the fact that the study of partial differential equations has been ongoing since the 18th Century, a large collection of information on the subject of the solution of such equations is available in the literature. The literature search reveals that whereas analytical solutions are well developed for idealized geometries and boundary conditions which can only be applicable to few practical problems, numerical schemes are more suited to solving partial differential equations that govern practically based problems. A discussion on the available numerical schemes in literature is provided, and special attention accorded to the Boundary element method since this is the basis on which the Green element method (the numerical scheme for the research work) is based.

If a solution can be developed for a partial differential governing equation, preference is to develop an analytical solution. But unfortunately, for many problems, it is extremely *difficult*, if not impossible, to get an analytic solution.

Reasons for the limitation of analytic solution techniques range from irregular geometry, through heterogeneity of the domain to the nonlinearity of the differential equation. Sometimes, the analytic solution may be too *complicated* and one would prefer to use an approximation. Numerical methods (methods in which approximate solutions are derived and coded on computers) have proved to be extremely useful in overcoming the limitations of analytical solution techniques and in addressing practically realistic problems that are on irregular geometries, heterogeneous, and nonlinear.

In solving partial differential equations using numerical techniques, the primary challenge is to create a set of relationships which approximates the equation of interest so that the solution is consistently reliable with respect to the element size and is numerically stable, implying that errors in the input data and intermediate calculations do not accumulate and cause the resulting output to be meaningless.

OBJECTIVES OF THE STUDY

The most immediate area of interest is the search for a reliable solution technique for the Time Dependent Coordinate Transformation algorithm in two dimensions. The main objective would be to either construct the mesh to allow for a square system of equations for both the nodal velocities x and the solution u or to deduce a reliable and more importantly stable method for finding a solution which will approximately satisfy all equations without creating escalating oscillations.

BOUNDARY AND INITIAL CONDITIONS

There are three types of boundary conditions. Defining a domain R, its boundary , the coordinates n and s normal (outward) and along the boundary, and functions f ; g on the boundary, the three boundary conditions are

- Dirichlet conditions with u = f on ∂R:
- Neumann conditions with ∂u= ∂n = f or ∂u= ∂s = g on ∂R:
- Mixed (Robin) conditions ∂u= ∂n+ku = f , k > 0, on ∂R.

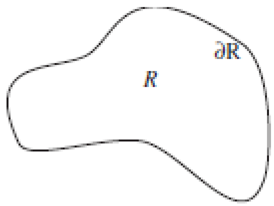

Figure : Simulation domain R with a boundary ∂R.

Dirichlet conditions can only be applied if the solution is known on the boundary and if f is analytic. These are frequently used for the flow (velocity) into a domain. More common are Neumann conditions.

Second order equations

Consider the PDE

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x \partial y}+C\frac{\partial^2 u}{\partial y^2}+H=0$$

As above we attempt to solve this equation through a total differential i.e.

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

Thus we define $P = \frac{\partial u}{\partial x}$ and $Q = \frac{\partial u}{\partial y}$ and determine the total differentials

dP = R dx + Sdy

dQ = S dx+T dy

- From the definition of P; Q we determine:

$$R = \frac{\partial^2 u}{\partial x^2}, \quad S = \frac{\partial^2 u}{\partial x \partial y}, \quad T = \frac{\partial^2 u}{\partial y^2}$$

- Evaluating

$$A\frac{dP}{dx} + C\frac{dQ}{dy} \text{ yields: } A\frac{\partial^2 u}{\partial x^2} + \left(A\frac{dy}{dx} + C\frac{dx}{dy}\right)\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2}$$

- On Comparison:

$$\left(A\frac{dy}{dx} + C\frac{dx}{dy}\right) - B = 0 \text{ or}$$

$$A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$$

$$\frac{dy}{dx} = \frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

The constant H can be considered in the solution for the total differentials. For the case of a time dependent compressible fluid without dissipation the governing set of PDE's is hyperbolic with two real characteristics. In this case the characteristics are the lines in space time along which information propagates. Considering the changes at a fixed location in a system over a small time interval implies that any changes are originating from the immediate vicinity of this point. In contrast any small change of the boundary conditions for an elliptic PDE effect the entire system.

Equation provides the curves along which one has to propagate the values of P and Q to find a solution to the PDE. This solution can be constructed from any boundary along which P, Q, and u are specified. To do so one has to solve the equations

$$\begin{aligned} du &= Pdx + Qdy \\ A\frac{dP}{dx} + C\frac{dQ}{dy} + H &= 0 \end{aligned}$$

or as finite differences

$$\begin{aligned} \Delta u &= P\Delta x + Q\Delta y \\ A\frac{\Delta P}{\Delta x} + C\frac{\Delta Q}{\Delta y} + H &= 0 \end{aligned}$$

For any intersection point Z of two characteristics close to a line where P, Q, and u are specified, the above equations provide two equations for DP and DQ from the boundary to the intersection.

Thus it is possible to compute the values of P and Q at the point of the intersection. Using P and Q one can compute Du and thus u at the point of the intersection. This method can be employed to construct the solution u to a hyperbolic PDE.

Coordinate transformations

An important question is whether a coordinate transformation can change the character of a differential equation. Obviously this should not be the case because the type of a PDE should be a characteristic property of a PDE with real implications (i.e., determine the type of solutions) and thus the type of a PDE ought to be invariant against coordinate transformation.

Assuming a coordinate transformation

$\xi(x,y), \eta(x,y)$ and $\tilde{u}(\xi(x,y), \eta(x,y)) = u(x,y)$ equation

$$A'\frac{\partial^2 \tilde{u}}{\partial \xi^2} + B'\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + C'\frac{\partial^2 \tilde{u}}{\partial \eta^2} + H' = 0$$

$$\begin{aligned} A' &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\ B' &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\ C' &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \end{aligned}$$

Multiple independent variables

Let us consider now a single dependent variable u but m independent variables. Similar to the second order differential equation is

$$\sum_{j=1}^{m}\sum_{k=1}^{m} a_{jk}\frac{\partial^2 u}{\partial x_j \partial x_k} + H = 0$$

Systems of equations - two dependent variables

Thus far we have considered only a single dependent variable. Frequently this is not the case such that one should consider methods for classification which can be applied to systems of PDE's. Let us first consider the case of

$$\begin{aligned} A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial u}{\partial y} + A_{12}\frac{\partial v}{\partial x} + B_{12}\frac{\partial v}{\partial y} &= 0 \\ A_{21}\frac{\partial u}{\partial x} + B_{21}\frac{\partial u}{\partial y} + A_{22}\frac{\partial v}{\partial x} + B_{22}\frac{\partial v}{\partial y} &= 0 \end{aligned}$$

Note that a single second order equation can be brought into a system of linear equations. In a compact form these equations can be expressed as

$$\underline{\underline{\mathbf{A}}} \cdot \frac{\partial \mathbf{q}}{\partial x} + \underline{\underline{\mathbf{B}}} \cdot \frac{\partial \mathbf{q}}{\partial y} = 0$$

$$with \quad \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}$$

We seek a solution again in terms of a total differential of the form

$$m_1 du + m_2 dv = m_1 \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + m_2 \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right).$$

Multiplying (3.9) with L_1 and (3.10) with L_2 we obtain for the coefficients of the derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$:

$$\begin{aligned} L_1 A_{11} + L_2 A_{21} &= m_1 dx \\ L_1 B_{11} + L_2 B_{21} &= m_1 dy \\ L_1 A_{12} + L_2 A_{22} &= m_2 dx \\ L_1 B_{12} + L_2 B_{22} &= m_2 dy \end{aligned}$$

Eliminating the rhs terms by multiplying these equation with dx or dy we obtain:

$$\begin{aligned} (L_1 A_{11} + L_2 A_{21}) dy &= (L_1 B_{11} + L_2 B_{21}) dx \\ (L_1 A_{12} + L_2 A_{22}) dy &= (L_1 B_{12} + L_2 B_{22}) dx \end{aligned}$$

or in vector form

$$\left(\underline{\underline{\mathbf{A}}}^t dy - \underline{\underline{\mathbf{B}}}^t dx \right) \cdot \mathbf{L} = 0$$

SCOPE OF THE STUDY

Despite the limited success of the Time Dependent Coordinate Transformation method to generate solutions in two dimensions this thesis has presented an interesting solution technique for problems in one dimension. It is obvious though that the method still needs further work and application to other types of problem to test its robustness and suitability for widespread application. The success of the mass monitor in both one-dimensional and radial coordinate cases suggests that a similar approach could possibly be implemented for the solution of hyperbolic conservation laws.

It has also become apparent that the one-dimensional work could be implemented using the arc-length monitor.

$$M(u) = \sqrt{1 + u_x^2}$$

A discrete approximation to the equi-distribution quantity θ(t) would in this case take the form

$$1 + u_x^2 = \theta(t)^2 \approx 1 + \left(\frac{u_{i+1} - u_i}{x_{i+1} - x_i} \right)^2.$$

which could be rewritten as

$$(x_{i+1} - x_i) \sqrt{(\theta(t)^2 - 1)} = u_{i+1} - u_i.$$

REFERENCES

- Solin, P. (2012), Partial Differential Equations and the Finite Element Method, Hoboken, NJ: J. Wiley & Sons, ISBN 0-471-72070-4.
- Solin, P.; Segeth, K. & Dolezel, I. (2013), Higher-Order Finite Element Methods, Boca Raton: Chapman & Hall/CRC Press, ISBN 1-58488-438-X.
- Stephani, H. (2011), Differential Equations: Their Solution Using Symmetries. Edited by M. MacCallum, Cambridge University Press.
- Wazwaz, Abdul-Majid (2012). Partial Differential Equations and Solitary Waves Theory. Higher Education Press. ISBN 90-5809-369-7.
- Zwillinger, D. (2013), Handbook of Differential Equations (3rd ed.), Boston: Academic Press, ISBN 0-12-784395-7.
- Gershenfeld, N. (2009), The Nature of Mathematical Modeling (1st ed.), New York: Cambridge University Press, New York, NY, USA, ISBN 0-521-57095-6.
- Krasil'shchik, I.S. & Vinogradov, A.M., Eds. (2009), Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, American Mathematical Society, Providence, Rhode Island,USA, ISBN 0-8218-0958-X.
- Krasil'shchik, I.S.; Lychagin, V.V. & Vinogradov, A.M. (2006), Geometry of Jet Spaces and Nonlinear Partial Differential Equations, Gordon and Breach Science Publishers, New York, London, Paris, Montreux, Tokyo, ISBN 2-88124-051-8.