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A Study on Partial Differential Equations and Their Scope

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Abstract – In mathematics, a partial differential equation(PDE) is a differential equation that contains unknown multivariable functions and their partial derivatives. (This is in contrast to ordinary differential equations (ODEs), which deal with functions of a single variable and their derivatives.) PDEs are used to formulate problems involving functions of several variables, and are either solved by hand, or used to create a relevant computer model.

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INTRODUCTION

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrostatics, electrodynamics, fluid flow, elasticity, or quantum mechanics. These seemingly distinct physical phenomena can be formalised similarly in terms of PDEs. Just as ordinary differential equations often model one-dimensional dynamical systems, partial differential equations often model multidimensional systems. PDEs find their generalisation in stochastic partial differential equations.

Partial differential equations (PDEs) are equations that involve rates of change with respect tocontinuous variables. The position of a rigid body is specified by six numbers, but the configuration of a fluid is given by the continuous distribution of several parameters, such as the temperature, pressure, and so forth. The dynamics for the rigid body take place in a finite-dimensional configuration space; the dynamics for the fluid occur in an infinite-dimensional configuration space. This distinction usually makes PDEs much harder to solve than ordinary differential equations (ODEs), but here again there will be simple solutions for linear problems. Classic domains where PDEs are used include acoustics, fluid flow, electrodynamics, and heat transfer.

A partial differential equation (PDE) for the function $u(x_1, \cdots, x_n)$ is an equation of the form

$$f\left(x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1 \partial x_1},\ldots,\frac{\partial^2 u}{\partial x_1 \partial x_n},\ldots\right)=0.$$

If *f* is a linear function of *u* and its derivatives, then the PDE is called linear. Common examples of linear PDEs include the heat equation, the wave equation, Laplace's equation, Helmholtz equation, Klein–Gordon equation, and Poisson's equation.

A relatively simple PDE is

$$\frac{\partial u}{\partial x}(x,y)=0.$$

This relation implies that the function *u*(*x*,*y*) is independent of *x*. However, the equation gives no information on the function's dependence on the variable *y*. Hence the general solution of this equation is

$$u(x,y)=f(y),$$

Where *f* is an arbitrary function of *y*. The analogous ordinary differential equation is

$$\frac{\mathrm{d}u}{\mathrm{d}x}(x)=0,$$

which has the solution

$$u(x)=c,$$

where *c* is any constant value.

These two examples illustrate that general solutions of ordinary differential equations (ODEs) involve arbitrary constants, but solutions of PDEs involve arbitrary functions.

A solution of a PDE is generally not unique; additional conditions must generally be specified on the boundary of the region where the solution is defined. For instance, in the simple example above,

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the function $f(y)$ can be determined if u is specified on the line $x = 0$.

EXISTENCE AND UNIQUENESS

Although the issue of existence and uniqueness of solutions of ordinary differential equations has a very satisfactory answer with the Picard–Lindel öf theorem, that is far from the case for partial differential equations. The Cauchy–Kowalevski theorem states that the Cauchy problem for any partial differential equation whose coefficients are analytic in the unknown function and its derivatives, has a locally unique analytic solution. Although this result might appear to settle the existence and uniqueness of solutions, there are examples of linear partial differential equations whose coefficients have derivatives of all orders (which are nevertheless not analytic) but which have no solutions at all. Even if the solution of a partial differential equation exists and is unique, it may nevertheless have undesirable properties. The mathematical study of these questions is usually in the more powerful context of weak solutions.

An example of pathological behavior is the sequence (depending upon n) of Cauchy problems for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

with boundary conditions

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\sin(nx)}{n},$$

where n is an integer. The derivative of u with respect to y approaches 0 uniformly in x as n increases, but the solution is

$$u(x,y) = \frac{\sinh(ny)\sin(nx)}{n^2}.$$

This solution approaches infinity if nx is not an integer multiple of π for any non-zero value of y. The Cauchy problem for the Laplace equation is called *ill-posed* or *not well posed*, since the solution does not depend continuously upon the data of the problem. Such ill-posed problems are not usually satisfactory for physical applications.

NOTATION

In PDEs, it is common to denote partial derivatives using subscripts. That is:

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Especially in physics, del or Nabla (∇) is often used to denote spatial derivatives, and $\dot{u}$ $\ddot{u}$ for time derivatives. For example, the wave equation can be written as

$$\ddot{u} = c^2 \nabla^2 u$$

or

$$\ddot{u} = c^2 \Delta u$$

where Δ is the Laplace operator.

EXAMPLES

Heat equation in one space dimension

The equation for conduction of heat in one dimension for a homogeneous body has

$$u_t = \alpha u_{xx}$$

where $u(t,x)$ is temperature, and α is a positive constant that describes the rate of diffusion. The Cauchy problem for this equation consists in specifying $u(0, x)= f(x)$, where $f(x)$ is an arbitrary function.

General solutions of the heat equation can be found by the method of separation of variables. Some examples appear in the heat equation article. They are examples of Fourier series for periodic f and Fourier transforms for non-periodic f. Using the Fourier transform, a general solution of the heat equation has the form

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-\alpha\xi^2 t} e^{i\xi x}\, d\xi,$$

where F is an arbitrary function. To satisfy the initial condition, F is given by the Fourier transform of f, that is

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

If f represents a very small but intense source of heat, then the preceding integral can be approximated by the delta distribution, multiplied by the strength of the source. For a source whose strength is normalized to 1, the result is

$$F(\xi) = \frac{1}{\sqrt{2\pi}},$$

and the resulting solution of the heat equation is

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\xi^2 t} e^{i\xi x} d\xi.$$

This is a Gaussian integral. It may be evaluated to obtain

$$u(t,x) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

This result corresponds to the normal probability density for x with mean 0 and variance $2\alpha t$. The heat equation and similar diffusion equations are useful tools to study random phenomena.

CLASSIFICATION

Some linear, second-order partial differential equations can be classified as parabolic, hyperbolic and elliptic. Others such as the Euler–Tricomi equation have different types in different regions. The classification provides a guide to appropriate initial and boundary conditions, and to smoothness of the solutions.

EQUATIONS OF FIRST ORDER

EQUATIONS OF SECOND ORDER

Assuming $u_{xy} = u_{yx}$, the general second-order PDE in two independent variables has the form

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots \text{(lower order terms)} = 0,$$

where the coefficients A, B, C etc. may depend upon x and y. If $A^2 + B^2 + C^2 > 0$ over a region of the xy plane, the PDE is second-order in that region. This form is analogous to the equation for a conic section:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

More precisely, replacing ∂_x by X, and likewise for other variables (formally this is done by aFourier transform), converts a constant-coefficient PDE into a polynomial of the same degree, with the top degree (a homogeneous polynomial, here a quadratic form) being most significant for the classification.

Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $B^2 - 4AC$, the same can be done for a second-order PDE at a given point. However, the discriminant in a PDE is given by $B^2 - AC$, due to the convention of the xy term being $2B$ rather than B; formally, the discriminant (of the associated quadratic form) is $(2B)^2 - 4AC = 4(B^2 - AC)$, with the factor of 4 dropped for simplicity.

$B^2 - AC < 0$: solutions of elliptic PDEs are as smooth as the coefficients allow, within the interior of the region where the equation and solutions are defined. For example, solutions of Laplace's equation are analytic within the domain where they are defined, but solutions may assume boundary values that are not smooth. The motion of a fluid at subsonic speeds can be approximated with elliptic PDEs, and the Euler–Tricomi equation is elliptic where $x < 0$.

$B^2 - AC = 0$: equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases. The Euler–Tricomi equation has parabolic type on the line where $x = 0$.

$B^2 - AC > 0$: hyperbolic equations retain any discontinuities of functions or derivatives in the initial data. An example is the wave equation. The motion of a fluid at supersonic speeds can be approximated with hyperbolic PDEs, and the Euler–Tricomi equation is hyperbolic where $x > 0$.

If there are n independent variables $x_1, x_2, \ldots, x_n$, a general linear partial differential equation of second order has the form

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower-order terms} = 0.$$

The classification depends upon the signature of the eigenvalues of the coefficient matrix $a_{i,j}$..

1. Elliptic: The eigenvalues are all positive or all negative.

2. Parabolic : The eigenvalues are all positive or all negative, save one that is zero.

3. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or

there is only one positive eigenvalue and all the rest are negative.

4. Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. There is only limited theory for ultrahyperbolic equations (Courant and Hilbert, 1962).

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