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A Study on Binomial Transform and its Significance

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Abstract – The binomial transform is a sequence transformation (i.e., a transform of a sequence) that computes its forward differences. It is closely related to the Euler transform, which is the result of applying the binomial transform to the sequence associated with its ordinary generating function.

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INTRODUCTION

The binomial transform, T, of a sequence, $\{a_n\}$, is the sequence $\{s_n\}$ defined by

$$s_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a_k.$$

Formally, one may write

$$s_n = (Ta)_n = \sum_{k=0}^{\infty} T_{nk} a_k$$

for the transformation, where T is an infinite-dimensional operator with matrix elements T_{nk}. The transform is an involution, that is,

$$TT = 1$$

or, using index notation,

$$\sum_{k=0}^{\infty} T_{nk} T_{km} = \delta_{nm}$$

where δ_{nm} is the Kronecker delta. The original series can be regained by

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k.$$

The binomial transform of a sequence is just the nth forward differences of the sequence, with odd differences carrying a negative sign, namely:

$$s_0 = a_0$$

$$s_1 = -(\Delta a)_0 = -a_1 + a_0$$

$$s_2 = (\Delta^2 a)_0 = -(-a_2 + a_1) + (-a_1 + a_0) = a_2 - 2a_1 + a_0$$

$$\vdots$$

$$s_n = (-1)^n (\Delta^n a)_0$$

where Δ is the forward difference operator.

Some authors define the binomial transform with an extra sign, so that it is not self-inverse:

$$t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_k$$

whose inverse is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} t_k.$$

Example

Binomial transforms can be seen in difference tables. Consider the following:

0		1		10		63		324		1485
	1		9		53		261		1161	
		8		44		208		900		
			36		164		692			
				128		528				
					400					

The top line 0, 1, 10, 63, 324, 1485,... (a sequence defined by $(2n^2 + n)3^{n-2}$) is the (non-involutive version of the) binomial transform of the diagonal 0, 1, 8, 36, 128, 400,... (a sequence defined by $n^2 2^{n-1}$).

The binomial transform is the shift operator for the Bell numbers. That is,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k$$

where the B_n are the Bell numbers.

ORDINARY GENERATING FUNCTION

The transform connects the generating functions associated with the series. For the ordinary generating function, let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

and

$$g(x) = \sum_{n=0}^{\infty} s_n x^n$$

then

$$g(x) = (Tf)(x) = \frac{1}{1-x} f\left(\frac{x}{x-1}\right).$$

EULER TRANSFORM

The relationship between the ordinary generating functions is sometimes called the Euler transform. It commonly makes its appearance in one of two different ways.

In one form, it is used to accelerate the convergence of an alternating series. That is, one has the identity

$$\sum_{n=0}^{\infty} (-1)^n a_n = \sum_{n=0}^{\infty} (-1)^n \frac{\Delta^n a_0}{2^{n+1}}$$

which is obtained by substituting x=1/2 into the last formula above.

The terms on the right hand side typically become much smaller, much more rapidly, thus allowing rapid numerical summation.

The Euler transform can be generalized (Borisov B. and Shkodrov V., 2007):

$$\sum_{n=0}^{\infty} (-1)^n \binom{n+p}{n} a_n = \sum_{n=0}^{\infty} (-1)^n \binom{n+p}{n} \frac{\Delta^n a_0}{2^{n+p+1}},$$

where p = 0, 1, 2,...

The Euler transform is also frequently applied to the Euler hyper-geometric integral ${}_2F_1$. Here, the Euler transform takes the form:

$${}_2F_1(a,b;c;z) = (1-z)^{-b} {}_2F_1\left(c-a, b; c; \frac{z}{z-1}\right).$$

The binomial transform, and its variation as the Euler transform, is notable for its connection to the continued fraction representation of a number. Let $0 < x < 1$ have the continued fraction representation

$$x = [0; a_1, a_2, a_3, \cdots]$$

then

$$\frac{x}{1-x} = [0; a_1 - 1, a_2, a_3, \cdots]$$

and

$$\frac{x}{1+x} = [0; a_1 + 1, a_2, a_3, \cdots].$$

EXPONENTIAL GENERATING FUNCTION

For the exponential generating function, let

$$\overline{f}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

and

$$\overline{g}(x) = \sum_{n=0}^{\infty} s_n \frac{x^n}{n!}$$

then

$$\overline{g}(x) = (T\overline{f})(x) = e^x \overline{f}(-x).$$

The Borel transform will convert the ordinary generating function to the exponential generating function.

INTEGRAL REPRESENTATION

When the sequence can be interpolated by a complex analytic function, then the binomial transform of the sequence can be represented by means of a Norlund–Rice integral on the interpolating function.

Prodinger gives a related, modular-like transformation: letting

$$u_n = \sum_{k=0}^{n} \binom{n}{k} a^k (-c)^{n-k} b_k$$

gives

$$U(x) = \frac{1}{cx+1} B\left(\frac{ax}{cx+1}\right)$$

where *U* and *B* are the ordinary generating functions associated with the series $\{u_n\}$ and $\{b_n\}$, respectively.

The rising *k*-binomial transform is sometimes defined as

$$\sum_{j=0}^{n} \binom{n}{j} j^k a_j.$$

The falling *k*-binomial transform is

$$\sum_{j=0}^{n} \binom{n}{j} j^{n-k} a_j.$$

Both are homomorphisms of the kernel of the Hankel transform of a series.

In the case where the binomial transform is defined as

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} a_i = b_n.$$

Let this be equal to the function $\mathfrak{J}(a)_n = b_n$.

If a new forward difference table is made and the first elements from each row of this table are taken to form a new sequence $\{b_n\}$, then the second binomial transform of the original sequence is,

$$\mathfrak{J}^2(a)_n = \sum_{i=0}^{n} (-2)^{n-i} \binom{n}{i} a_i.$$

If the same process is repeated *k* times, then it follows that,

$$\mathfrak{J}^k(a)_n = b_n = \sum_{i=0}^{n} (-k)^{n-i} \binom{n}{i} a_i.$$

Its inverse is,

$$\mathfrak{J}^{-k}(b)_n = a_n = \sum_{i=0}^{n} k^{n-i} \binom{n}{i} b_i.$$

This can be generalized as,

$$\mathfrak{J}^k(a)_n = b_n = (\mathbf{E} - k)^n a_0$$

where $\mathbf{E}$ is the shift operator.

Its inverse is

$$\mathfrak{J}^{-k}(b)_n = a_n = (\mathbf{E} + k)^n b_0.$$

We restrict ourselves to the zero'th order modified Hankel transform, since we have shown in the previous section how higher order modified Hankel transforms can be reduced to the zero'th order transform.

To stress that no exponential sampling is needed, we start by sampling the objective function f(t) on a linear grid with step Δ, yielding the sample set {f(kΔ)}. We then reconstruct the function f(t) by linear interpolation as

$$f(t) = \sum_{k=0}^{r-1} f(k\Delta)\phi\left(\frac{t}{\Delta} - k\right) + \epsilon_T(t) + \epsilon_I(t)$$

where $\phi(t)$ is the linear interpolatory kernel, also known as the hat function,

$$\phi(t) = (1 - |t|)\Upsilon(1 - |t|)$$

and $\epsilon_T(t)$, $\epsilon_I(t)$ are respectively the truncation and interpolation errors

$$\epsilon_T(t) = \sum_{k=r}^{\infty} f(k\Delta)\phi\left(\frac{t}{\Delta} - k\right)$$

$$\epsilon_I(t) = f(t) - \sum_{k=0}^{\infty} f(k\Delta)\phi\left(\frac{t}{\Delta} - k\right)$$

The L_2 norm of the truncation error satisfies

$$\|\epsilon_T\| = \sqrt{\int |\epsilon_T(t)|^2 dt} \leq \sqrt{2\Delta/3} \sum_{k=r}^{\infty} |f(k\Delta)|$$

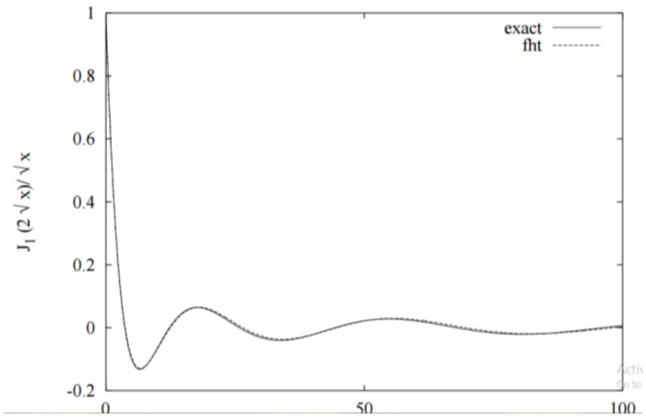
since kφk = p 2/3. Hence the truncation error is small provided |f(t)| has a fastly decreasing tail for t ≥ rΔ. Note that in general k^2T k → 0 for r → ∞, provided supt |f(t)|t η < ∞ for some η > 1. The interpolation error mainly depends on the smoothness of the function f(t) and the quasi-interpolant character of the kernel φ(t).

$$\|\epsilon_I\| \leq C\Delta^q \|f^{(q)}\|$$

provided f(t) has its qth derivative in L2[0, ∞] and provided the interpolation kernel is a quasiinterpolant of order q, i.e.

$$\sum_{k \in Z} k^m \phi(x-k) = x^m \qquad m = 0, \ldots, q-1$$

This is the case for the linear interpolatory kernel φ(t) for which q = 2. Note that in general $k^2 Ik \to 0$ for $\Delta \to 0$. Since the zero'th order modified Hankel transform is unitary, the truncation and interpolation errors propagate through the transform process with their L2 norms unchanged, and hence we can as well omit the error terms.



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