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# AN ANALYSIS OF TIME DEPENDENT COORDINATE TRANSFORMATION FOR SECOND ORDER PARTIAL DIFFERENTIAL EQUATION

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# An analysis of Time Dependent Coordinate Transformation for Second Order Partial Differential Equation

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***Abstract – Time-dependent problems that are modeled by initial-boundary value problems for parabolic or hyperbolic partial differential equations can be treated with the boundary integral equation method. The ideal situation is when the right-hand side in the partial differential equation and the initial conditions vanish, the data are given only on the boundary of the domain, the equation has constant coefficients, and the domain does not depend on time.***

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## INTRODUCTION

In this situation, the transformation of the problem to a boundary integral equation follows the same well-known lines as for the case of stationary or time-harmonic problems modeled by elliptic boundary value problems. The same main advantages of the reduction to the boundary prevail: Reduction of the dimension by one, and reduction of an unbounded exterior domain to a bounded boundary.

There are, however, specific difficulties due to the additional time dimension: Apart from the practical problems of increased complexity related to the higher dimension, there can appear new stability problems. In the stationary case, one often has unconditional stability for reasonable approximation methods, and this stability is closely related to variational formulations based on the ellipticity of the underlying boundary value problem.

In the time dependent case, instabilities have been observed in practice, but due to the absence of ellipticity, the stability analysis is more difficult and fewer theoretical results are available.

Like stationary or time-harmonic problems, transient problems can be solved by the boundary integral equation method. When the material coefficients are constant, a fundamental solution is known and the data are given on the boundary, the reduction to the boundary provides efficient numerical methods in particular for problems posed on unbounded domains.

Such methods are widely and successfully being used for numerically modeling problems in heat conduction and diffusion, in the propagation and scattering of acoustic, electromagnetic and elastic waves, and in fluid dynamics. One can distinguish three approaches to the application of boundary integral methods on parabolic and hyperbolic initial-boundary value problems: Space-time integral equations, Laplace-transform methods, and time-stepping methods.

Analysis of variational methods exists for the main domains of application of space-time boundary integral equations: First of all for the scalar wave equation, where the boundary integrals are given by retarded potentials, but also for elastodynamics (Becache, 1993; Becache and Ha-Duong, 1994; Chudinovich, 1993c; Chudinovich, 1993b; Chudinovich, 1993a), piezoelectricity (Khutoryansky and Sosa, 1995), and for electrodynamics (Bachelot and Lange, 1995; Bachelot et al., 2001; Rynne, 1999; Chudinovich, 1997).

An extensive review of results on variational methods for the retarded potential integral equations is given by HaDuong (2003). As in the parabolic case, collocation methods are practically important for the hyperbolic spacetime integral equations. For the retarded potential integral equation, the stability and convergence of collocation methods has now been established (Davies, 1994; Davies, 1998; Davies and Duncan, 1997; Davies and Duncan, 2003). Finally, let us mention that there have also been important developments in the field of fast methods for space-time boundary integral equations (Michielssen, 1998; Jiao et al., 2002; Michielssen et al., 2000; Greengard and Strain, 1990; Greengard and Lin, 2000).

For the description of the general principles, we consider only the simplest model problem of each type. We also assume that the right hand sides have the right structure for the application of a "pure" boundary integral method: The volume sources and the initial conditions vanish, so that the whole system is driven by boundary sources.

**Elliptic problem** (Helmholtz equation with frequency $\omega \in \mathbb{C}$):

$$(\Delta + \omega^2)u = 0 \quad \text{in } \Omega$$
$$u = g \ \text{(Dirichlet)} \quad \text{or } \partial_n u = h \ \text{(Neumann)} \quad \text{on } \Gamma$$
radiation condition at $\infty$

**Parabolic problem** (heat equation):

$$(\partial_t - \Delta)u = 0 \quad \text{in } Q$$
$$u = g \ \text{(Dirichlet)} \quad \text{or } \partial_n u = h \ \text{(Neumann)} \quad \text{on } \Sigma$$
$$u = 0 \quad \text{for } t \le 0$$

**Hyperbolic problem** (wave equation with velocity $c > 0$):

$$(c^{-2}\partial_t^2 - \Delta)u = 0 \quad \text{in } Q$$
$$u = g \ \text{(Dirichlet)} \quad \text{or } \partial_n u = h \ \text{(Neumann)} \quad \text{on } \Sigma$$
$$u = 0 \quad \text{for } t \le 0$$

The derivation of boundary integral equations follows from a general method that is valid (under suitable smoothness hypotheses on the data) in the same way for all 3 types of problems. In fact, what counts for (P) and (H) is the property that the lateral boundary Σ is non-characteristic. The first ingredient for a BEM is a fundamental solution G. As an example, in 3 dimensions we have, respectively

$$G_\omega(x) = \frac{e^{i\omega|x|}}{4\pi|x|}$$

$$G(t,x) = \begin{cases} (4\pi t)^{-3/2} e^{-\frac{|x|^2}{4t}} & (t \ge 0) \\ 0 & (t \le 0) \end{cases}$$

$$G(t,x) = \frac{1}{4\pi|x|}\,\delta\!\left(t - \frac{|x|}{c}\right)$$

Representation formulas for a solution u of the homogeneous partial differential equation and x 6∈ Γ are obtained from Green's formula, applied with respect to the space variables in the interior and exterior domain. We assume that u is smooth in the interior and the exterior up to the boundary, but has a jump across the boundary.

## REVIEW OF LITERATURE

This objective reality consisted in tracing the radius vectors with light signals. Hence, in despite their strong appearance of mathematical tricks, the manipulations were not tricks at all. The derivation of the Lorentz transformation was correct.

Second, he "did not seem to be reasoning at all". He discarded the concepts of absolute rest and absolute motion but described in detail a thought experiment which seems to be the only one enabling the 'blind' inertial observers to determine absolute speeds in their reference frames. He proposed the experiment for deducing the Lorentz transformation in the idea that identical inertial clocks would run at rates depending on their speed. But, because he did not realize the role plaid by the light signals in this experiment, needed to manipulate some equations to this end.

Unfortunately, he did not investigate further the diagram describing the experiment to see that this diagram actually validates 'abstract' coordinate systems at absolute rest in his theory. There becomes evident that Einstein was not aware that by light signals has specified the time changing magnitude and direction of the radius vectors of geometrical points moving with respect to inertial observers (which should lead him to the LT as a complementary time-dependent coordinate transformation) but he used light signals,) the graphical addition of travel times as scalar quantities needed be developed in his theory but he worked only with light signals tracing abscissas of geometrical points and dropped the square of β in his equations linear in , according to the graphical addition of travel times as scalar quantities, the equation assured the independence of the linear equations in β (making them a coordinate transformation) but he took into account this equation in order to obtain the "addition theorem for speeds" and the coordinate system at absolute rest plays an essential role in his theory but he consecrated a version of the light-speed principle that saved his theory from the inconsistencies raised by the arbitrary removal of the coordinate system at absolute rest.

It is as if Einstein reconstituted by flashes on the derivation of the LT as a complementary time-dependent coordinate transformation that pre-existed in his subconscious. The correctness of all the manipulations of equations supports the revealed knowledge of the original paper.

Dirac and der Waerden should obtain genuine subquantum information. The application of this information (disclosed further in this book to radically new technologies (that should happen as early as by the 1940's) gives the real dimension of the impact which the missed and distortedly perceived revealed knowledge had (still has) upon the advancement of physics, finally upon the progress of the mankind.

A Second Order Partial Differential Equation is a type of second-order partial differential equation, describing a wide family of problems in science including heat diffusion, ocean acoustic propagation, and stock option pricing. These problems, also known as evolution problems, describe physical or mathematical systems with a time variable, and which behave essentially like heat diffusing through a solid.

A partial differential equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + \cdots = 0$$

is parabolic if it satisfies the condition

$$B^2 - 4AC = 0$$

A simple example of a parabolic PDE is the one-dimensional heat equation, where u(t,x) is the temperature at time t and at position x, and k is a constant. The symbol ut signifies the partial derivative with respect to the time variable t, and similarly uxx is the second partial derivative with respect to x.

Later workers such as Morgan (1952), Hansen (1964), Krzywoblocki (1963) and Wecker and Hayes (1960) investigated similarity methods by considering the governing equations first and only examining the boundary and initial conditions as a later step, if at cell. Another group of workers developed similarity methods by starting with a complete mathematical formulation and thus motivated to examine less complete (and more general) problems, see for example Coles (1962), Abbott and Kline (1960) and Gukhman (1965).

## RESEARCH STUDY

An examination of these earlier works show that the initial problem statement as far as assumed completeness determined to a large extent the kind of mathematical approach employed. The more information that was known, the more direct was the method developed for finding a similarity solution and at the same time, the less general were both the methods and the conclusions ( as regards "general solutions" ).

It is not suggested that this dichotomy is necessarily bad. The more general techniques, such as group theory methods, have produced powerful theorems and yield results with a minimum of mathematical busy-work. On the other hand, the group theory methods are difficult for the average engineer to follow because their motivation is mathematical, not physical and this has inhibited their wide use. Also, somewhat amazingly, the more powerful mathematical techniques have been to a degree more restrictive in some of their aspects (such as the "class of assumed transformation") than the less elegant methods.

We implement Reduced Differential Transform Method to approximately solve the nonlinear dispersive $K(m, n)$ type equations. This method is an alternative approach to overcome the demerit of complex calculation of differential transform method, capable of reducing the size of calculation and easily overcomes the difficulty of the perturbation technique or Adomian polynomials. To illustrate the application of this method, the two special cases, $K(2, 2)$ and $K(3, 3)$ are discussed.

We study the existence of radially symmetric solitary waves for a non-linear Schrodinger-Poisson system. In contrast to all previous results, we consider the presence of a positive potential, of interest in physical applications.

We introduce a new dispersion-velocity particle method for approximating solutions of linear and nonlinear dispersive equations. This is the first time in which particle methods are being used for solving such equations.

We numerically test our new method for a variety of linear and nonlinear problems. In particular we are interested in nonlinear equations which generate structures that have non-smooth fronts. It is remarkable to see that our particle method is capable of capturing the nonlinear regime of a compacton-cotnpacton type interaction.

We introduce a new dispersion-velocity particle method for approximating solutions of linear and nonlinear dispersive equations. This is the first time in which particle methods are being used for solving such equations.

Our method is based on an extension of the diffusion-velocity method of Degond and Mustieles (SIAM J. Sci. Stat. Comput. 11(2), 293 (1990)) to the dispersive framework. The main analytical result we provide is the short time existence and uniqueness of a solution to the resulting dispersion-velocity transport equation.

We numerically test our new method for a variety of linear and nonlinear problems. In particular we are interested in nonlinear equations which generate structures that have non-smooth fronts. Our simulations show that this particle method is capable of capturing the nonlinear regime of a compacton–compacton type interaction.

A partial differential equation (PDE) is called dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider $iu_t + u_{xx} = 0$. If we try a simple wave of the form $u(x,t) = Ae^{i(kx-\omega t)}$, we see that it satisfies the equation if and only if $\omega = k^2$. This is called the dispersive relation and shows that the frequency is a real valued function of the wave number.

If we denote the phase velocity by $v = \frac{\omega}{k}$ we can write the solution as $u(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave propagates in such a way that large wave numbers travel faster than smaller ones. (Trying a wave solution of the same form to the heat equation

$u_t - u_{xx} = 0$, we obtain that the lj is complexed valued and the wave solution decays exponential in time.

On the other hand the transport equation $u_t - u_x = 0$ and the one dimensional wave equation $u_{tt} = u_{xx}$ are traveling waves with constant velocity.)

If we add nonlinear effects and study $iu_t + u_{xx} = f(u)$, we will see that even the existence of solutions over small times requires delicate techniques.

Going back to the linear equation, consider $u_0(x) = \int_R \hat{u}_0(k) e^{ikx} dk$. For each fixed k the wave solution becomes $u(x,t) = \hat{u}_0(k) e^{ik(x-kt)} = \hat{u}_0(k) e^{ikx} e^{-ik^2}$. Summing over k (integrating) we obtain the solution to our problem $u(x,t) = \int_R \hat{u}_0(k) e^{ikx - ik^2 t} dk$

Since $|\hat{u}(k,t)| = |\hat{u}_0(k)|$ we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$.

Thus the conservation of the $L^2$ norm (mass conservation or total probability) and the fact that high frequencies travel faster, leads to the conclusion that not only the solution will disperse into separate waves but that its amplitude will decay over time.

This is not anymore the case for solutions over compact domains. The dispersion is limited and for the nonlinear dispersive problems we notice a migration from low to high frequencies.

This fact is captured by zooming more closely in the Sobolev norm

$$\|u\|H^8 = \sqrt{\int |\hat{u}(k)|^2 (1 + |k|)^{2s} dk}$$

and observing that it actually grows over time. To analyze further the properties of dispersive PDEs and outline some recent developments we start with a concrete example.

However, in the non-relativistic limit regime, i.e. if $0 < \varepsilon \ll 1$ or the speed of light goes to infinity, the analysis and efficient computation of the KG equation are mathematically rather complicated issues.

## ANALYSIS AND INTERPRETATION

A different approach in which particle methods were used for approximating solutions of the heat equation and related models (such as the Fokker-Planck equation and a Boltzmann-like equation: the Kac equation), was introduced by Russo (2003).

In these works, the diffusion of the particles was described as a deterministic process in terms of a mean motion with a speed equal to the osmotic velocity associated with the diffusion process.

In a following work, the method was shown to be successful for approximating solutions to the two-dimensional Navier-Stokes (NS) equation in an unbounded domain. In this setup, the particles were convected according to the velocity field while their weights evolved according to the diffusion term in the vorticity formulation of the NS equations.

Another deterministic approach for approximating solutions of the parabolic equations with particle methods was introduced by Degond and Mustieles (2000). Their so-called diffusion-velocity method was based on defining the convective field associated with the heat operator which then allowed the particles to convect in a standard way.

For example, the one-dimensional heat equation $u_t = u_{xx}$ is rewritten as $u_t + (a(u)u)_x = 0$, where the velocity a(u) is taken as –ux/u. Particles carrying fixed masses will be then convicted with speed a(u). The convergence properties of the diffusion-velocity method were investigated, where short time existence and uniqueness of solutions for the resulting diffusion-velocity transport equation were proved.

The diffusion velocity method serves as the basic tool for the derivation of our particle methods in the dispersive world.

We focus our attention on linear and nonlinear dispersive partial differential equations.

Our model problem in the linear setup is the linear Airy equation,

$$u_t = u_{xxx}$$

The success of particle methods in approximating the oscillatory solutions that develop in this dispersive equation, provide us with valuable insight regarding the potential embedded in our approach.

In the nonlinear setup, we focus on equations which generate compactly supported solutions with non-smooth fronts, the prototype being the K( m, n) equation, which was introduced by Rosenau and Hyman (2003).

In this equation, a nonlinear dispersion term replaces the nonlinear dispersion term in the Korteweg-de Vries (KdV) equation, resulting with ·

$$K(m,n): \quad u_t + (u^m)_x + (u^n)_{xxx} = 0, \qquad m > 0, \qquad 1 < n \leq 3$$

For certain values of m and n, the K(m, n) equation has solitary waves which are compactly supported.

In particular, the variant K(2, 2),

$$K(2,2): \quad u_t + (u^2)_x + (u^2)_{xxx} = 0$$

has a fundamental "compact on" solution of the form

$$u(x,t) = \frac{4\lambda}{3}\left[\cos\left(\frac{x-\lambda t}{4}\right)\right]^2, \quad |x-\lambda t| \le 2\pi$$

After the first appearance of the compactions, it turned out that similar structures emerge as solutions for a much larger class of nonlinear PDEs, among which is, e.g.,

$$u_t + (u^m)_x + (u(u^n)_{xx})_x = 0, \quad m > 1, \quad m = m + 1$$

which we consider with m = 2, n = 1 as our non-linear model problem.

In this work, we are mainly interested in developing tools for approximating numerically solutions of equations which generate non-smooth structures.

Due to the discontinuity in the derivatives on the fronts of these emerging structures, standard numerical methods such as finite-differences and pseudo-spectral methods generate spurious oscillations on the fronts.

Moreover, in cases where a positive solution should remain positive in time; the spurious numerical oscillations might cause the solution to change sign. In this case, one can fall into an ill-posed region of the equation, and the numerical solution will cease to represent the solution of the equation at hand.

There have been several attempts in the literature .to address the complex numerical issues. For example, solutions of the compacton equation, K(2, 2), were obtained with finite-difference methods. In (de Frutos J., 2005), these finite-difference methods were shown to generate instabilities on the discontinuous fronts, which were interpreted there as shocks.

## CONCLUSION AND SUGGESTIONS

The existence theory in ID was given in and the analysis in 2D was recently announced in. Another interesting problem is the existence and uniqueness of the ground states, i.e. the solutions which minimize the total energy functional under the normalization constraint.

For the most simple-looking equation, i.e. the SN equation without external potential, the existence of a unique spherically symmetric ground state in 3D was proven by Lieb and in any dimension $d \le 6$ was given.

There is no global minimum of the energy functional for the repulsive SP equation without external potential since the infimum of its energy is always zero. When the Slater term is considered and in the absence of any external potential, the existence analysis of ground states in 3D was given in, and in particular the existence of a unique spherically symmetric ground state is proven in for the attractive case. To our knowledge, so far the existence analysis of higher bound states remains open.

Along the numerical front, self-consistent solutions of the SPS equation are important in the simulations of a quantum system. For example, time-independent SP equation was solved in for the eigenstates of the quantum system, and time-dependent spherically symmetric SP equation was considered in and time-dependent SN equation was treated in with three kinds of symmetry: spherical, axial and translational symmetry.

Most of the previous work applies Crank Nicholson time integration and finite difference for space discretization. Also, note that in general the ground states of the SPS equation will lose the symmetric profile due to the external potential and therefore one cannot obtain a reduced quasi-ID model as for the SN system, by studying which the SN equation was extensively investigated in. On the other hand, the computation of stationary states and dynamics of the NLS equation without Hartree potential has been extensively studied. Among the numerical methods proposed in the literature, discretizations based on a gradient flow with discrete normalization (GFDN) show more efficient in finding the ground and excited states of NLS modeling the Bose-Einstein condensates (BEC).

For dynamics, a time-splitting pseudospectral discretization shows its accuracy and efficiency in practice. Such results suggest that we can extend these successful tools to the computation of ground states and dynamics of the SPS equation. For example, similar methods were extended in to treat a Gross-Pitaevskii-Poisson type system which is used to model dipolar BEC, and a time-splitting approach was used in for computing the dynamics of the SPS equation with periodic boundary conditions in all space dimensions. However, there still remains an issue that how to approximate the Hartree potential properly, which definitely affects the overall accuracy and efficiency.

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