Journal of Advances in Science and Technology

Vol. IX, Issue No. XIX, May-2015, ISSN 2230-9659

AN ANALYSIS UPON SURJECTIVITY OF PARTIAL DIFFERENTIAL OPERATORS: FUNDAMENTAL SOLUTION TO THE DIVISION PROBLEM

AN INTERNATIONALLY INDEXED PEER REVIEWED & REFEREED JOURNAL

An Analysis upon Surjectivity of Partial Differential Operators: Fundamental Solution to the Division Problem

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Abstract – We give a sufficient condition for the surjectivity of partial differential operators with constant coefficients on a class of distributions on $\mathbb{R}^{n+1}$ ***(here we think of there being n space directions and one time direction), that are periodic in the spatial directions and tempered in the time direction.***

By proving a topological Paley-Wiener Theorem for Riemannian symmetric spaces of non-compact type, we show that a non-zero invariant differential operator is a homeomorphism from the space of test functions onto its image and hence surjective when extended to the space of distributions.

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INTRODUCTION

An important milestone in the general theory of partial differential equations is the solution to the division problem: Let *D* be a nonzero partial differential operator with constant coefficients and *T* be a distribution; can one find a distribution *S* such that *DS* = T? That this is always possible was established by Ehrenpreis (1956). See also Peter Wagner(2009) for the solution of various avatars of the division problem for different spaces of distributions.

In this article, we study the division problem in spaces of distributions on $\mathbb{R}^{n+1}$(where we think of there being n space directions and one time direction) that are periodic in the spatial directions. The study of such solution spaces arises naturally in control theory when one considers the so-called "spatially invariant systems". In the "behavioral approach" to control theory for such spatially invariant systems, a fundamental question is whether this class of distributions is an injective module over the ring of partial differential operators with constant coefficients. In light of this, one can first ask what happens with the division problem. Thus besides being a purely mathematical question that fits in the classical theme mentioned in the previous paragraph, there is also a behavioral control theoretic motivation for studying the division problem for distributions that are periodic in the spatial directions. Upon taking Fourier transform with respect to the spatial variables, the problem amounts to the following.

Problem: For which $P(\tau,\xi) \in \mathbb{C}[\tau,\xi_1,\cdots,\xi_n]$ is $P\left(\frac{d}{dt}, i\xi\right) : X \to X$ surjective, where

$$X = \{(T_\xi) = (T_\xi)_{\xi\in\mathbb{Z}^n} \in \mathcal{D}'(\mathbb{R})^{\mathbb{Z}^n} : \forall\varphi \in \mathcal{D}(\mathbb{R}),\ \exists k \in \mathbb{N}$$

$$: \forall\xi \in \mathbb{Z}^n,\ |\langle\varphi, T_\xi\rangle| \le k\cdot(1+|\xi|)^k\}.$$

($|\cdot|$denotes the 1-norm in $\mathbb{R}^n$.) An obvious necessary condition is that for all $\xi \in \mathbb{Z}^n,\ P\left(\frac{d}{dt}, i\xi\right) \not\equiv 0.$

However, that this condition is not sufficient is demonstrated by considering the following example.

Example 1. Let $c = \sum_{j=1}^{\infty} \frac{1}{2^{j!}}$ (a "Liouville number"). With $p_k := \sum_{j=1}^{k} 2^{k!-j!}$, $q_k := 2^{k!}$,

$$\left|c - \frac{p_k}{q_k}\right| = \sum_{j=k+1}^{\infty} \frac{1}{2^{j!}} = \frac{1}{2^{(k+1)!}} + \frac{1}{2^{(k+2)!}} + \cdots$$

$$< 2\cdot\frac{1}{2^{(k+1)!}} \le \left(\frac{1}{2^{k!}}\right)^k = \frac{1}{q_k^k},\quad k \in \mathbb{N}.$$

Now take $P = \xi_1 + c\xi_2$. Then $(1)_{\xi\in\mathbb{Z}^2}$ does not belong to the range of *P* because

$$\frac{1}{\xi_1 + c\xi_2} \notin X.$$

Indeed, otherwise there would exist an m such that

$$\frac{1}{|\xi_1 + c\xi_2|} \leq m(1 + |\xi_1| + |\xi_2|)^m \text{ for all } \xi_1, \xi_2 \in \mathbb{Z},$$

and in particular, with $\xi_1 = -p_k$, and $\xi_2 = q_k,\ k \in \mathbb{N}$,

$$q_k^{k-1} \leq \frac{1}{|\xi_1 + c\xi_2|} \leq m(1 + p_k + q_k|)^m$$

$$\leq m(kq_k + kq_k + kq_k)^m = m(3kq_k)^m,$$

a contradiction.

We consider a simpler situation and set

$$Y = \{(T_\xi) \in \mathcal{S}'(\mathbb{R})^{\mathbb{Z}^n} :\ \forall\varphi \in \mathcal{S}(\mathbb{R}),$$

$$\exists k \in \mathbb{N}:\ \forall\xi \in \mathbb{Z}^n,\ |\langle\varphi, T_\xi\rangle| \leq k\cdot(1 + |\xi|)^k\}$$

Our main result is the following:

Theorem 1. *Let* $P(\tau, i\xi) \in \mathbb{C}[\tau,\xi] = \mathbb{C}[\xi][\tau]$, *and for* $\xi \in \mathbb{Z}^n$,

$$P(\tau, i\xi) = c_\xi \cdot \prod_{j=1}^{m_\xi}(\tau - \lambda_{j,\xi}),$$

With $m_\xi \in \mathbb{N}_0,\ c_\xi \in \mathbb{C}\setminus\{0\},\ \lambda_{1,\xi},\cdots,\lambda_{m_\xi,\xi} \in \mathbb{C}$.

(The roots $\lambda_{j,\xi}$ *are arbitrarily arranged.)*

Let

$$d_\xi := \begin{cases} 1 & \text{if for all } j = 1,\cdots,m_\xi,\ \mathrm{Re}(\lambda_{j,\xi}) = 0, \\ \min\limits_{j:\mathrm{Re}(\lambda_{j,\xi})\neq 0} |\mathrm{Re}(\lambda_{j,\xi})| & \text{otherwise.} \end{cases}$$

If

$(1)\ (c_\xi^{-1}) \in s'(\mathbb{Z}^n)$ *and*

$(2)\ (d_\xi^{-1}) \in s'(\mathbb{Z}^n)$

then . $P\left(\frac{d}{dt}, i\xi\right) : Y \to Y$ *is surjective.*

From Example 1, it follows that the first condition is not superfluous. Here is an example demonstrating that the second condition is not superfluous either.

Example 2. Take $\left(\frac{d}{dt}, i\xi\right) := \frac{d}{dt} + \xi_1 + c\xi_2$

with the same *c* as in Example 1. and $T_\xi := 1,\ \xi \in \mathbb{Z}^2$.

Then $(S_\xi) = \left(\frac{1}{\xi_1 + c\xi_2}\right) \notin Y.$

PRELIMINARIES

There holds that

$$\begin{array}{ccccc} Y & \simeq & \mathcal{L}(\mathcal{S}(\mathbb{R}), s'(\mathbb{Z}^n)) & \simeq & \mathcal{S}'(\mathbb{R})\widehat{\otimes}s'(\mathbb{Z}^n). \\ (T_\xi) & \longmapsto & (\varphi \mapsto \langle\varphi, T_\xi\rangle) & & \end{array}$$

(For the first isomorphism we use the Closed Graph Theorem, while the second isomorphism follows,

$$\mathcal{S}'(\mathbb{R})\widehat{\otimes}s'(\mathbb{Z}^n) \simeq \mathcal{S}'(\mathbb{R})\widehat{\otimes}\mathcal{S}'((\mathbb{R}/\mathbb{Z})^n) \simeq \mathcal{S}'(\mathbb{R}\times(\mathbb{R}/\mathbb{Z})^n) = \widehat{Y},$$

that is, $\widehat{Y}$ is the dual of a Frechet space and there holds that for all $\widehat{T} \in \widehat{Y}$ there exists $k \in \mathbb{N}$ such that for all $\psi \in \mathcal{S}(\mathbb{R}\times(\mathbb{R}/\mathbb{Z})^n)$,

$$|\langle\psi, \widehat{T}\rangle| \leq k \cdot \sum_{|\alpha|\leq k} \|(1+|t|)^k\partial^\alpha\psi\|_\infty$$

Hence it follows that in Y:

for all $(T_\xi) \in Y$ there exists $k \in \mathbb{N}$ such that for all $\varphi \in \mathcal{S}(\mathbb{R})$ and all $\xi \in \mathbb{Z}^n$,

$$|\langle\varphi, T_\xi\rangle| \leq k\cdot(1+|\xi|)^k \cdot \sum_{j=0}^{k} \|(1+|t|)^k\varphi^{(j)}\|_\infty.$$

In particular for $\varphi \in \mathcal{S}(\mathbb{R})$ and $t \in \mathbb{R}$,

$$|(\varphi * \check{T}_\xi)(t)| \quad = \quad |\langle(\tau \mapsto \varphi(t+\tau)), T_\xi\rangle|$$

$$\leq k\cdot(1+|\xi|)^k \cdot \sum_{j=0}^{k} \left\|\tau \mapsto \left((1+|t-\tau|)^k\varphi^{(j)}(\tau)\right)\right\|_\infty$$

$$\leq \quad C_{\varphi,k}\cdot(1+|\xi|)^k\cdot(1+|t|)^k.$$

We will also need the following lemma.

Lemma 1. *Consider a monic polynomial*

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$P = \tau^d + c_{d-1,\xi}\tau^{d-1} + \cdots + c_{1,\xi}\tau + c_{0,\xi} \in s'(\mathbb{Z}^n)[\tau].$

For $\xi \in \mathbb{Z}^n$, *we factorize*

$$P(\tau, i\xi) = \prod_{j=1}^{d}(\tau - \lambda_{j,\xi}),$$

with $\lambda_{1,\xi}, \cdots, \lambda_{d,\xi} \in \mathbb{C}$. *(The roots* $\lambda_{j,\xi}$ *are arbitrarily arranged.) Then* $(\lambda_{j,\xi})_\xi \in s'(\mathbb{Z}^n)$, $j = 1, \cdots, d$.

Proof.

Let $(\tau+\alpha_\xi)(\tau^{d-1}+b_{d-2,\xi}\tau^{d-2}+\cdots+b_{1,\xi}\tau+b_{0,\xi}) = P$.

Then by comparing coefficients of the powers of τ, we obtain

$$\begin{aligned} \alpha_\xi \cdot b_{0,\xi} &= c_{0,\xi}, \\ \alpha_\xi \cdot b_{1,\xi} + b_{0,\xi} &= c_{1,\xi}, \\ &\vdots \\ \alpha_\xi \cdot b_{d-2,\xi} + b_{d-3,\xi} &= c_{d-2,\xi}, \\ \alpha_\xi + b_{d-2,\xi} &= c_{d-1,\xi}. \end{aligned}$$

We first show that $(\alpha_\xi) \in s'(\mathbb{Z}^n)$. Suppose this is not true. Then there exists a sequence $(\xi_k)_k$ such that

$$\lim_{k\to\infty} |\xi_k| = \infty$$

and $|\alpha_{\xi_k}| > k(1+|\xi_k|)^k$ for all k. But then from the first equation in the above equation array, it follows that

$$|b_{0,\xi_k}| = \frac{|c_{0,\xi_k}|}{|\alpha_{\xi_k}|} \overset{k\to\infty}{\longrightarrow} 0.$$

Then from the second equation in the above equation array, we also obtain that

$$|b_{1,\xi_k}| = \frac{|c_{1,\xi_k} - b_{0,\xi_k}|}{|\alpha_{\xi_k}|} \overset{k\to\infty}{\longrightarrow} 0.$$

Proceeding in this manner, we get eventually that

$$1 \le \frac{|c_{d-1,\xi_k} - b_{d-2,\xi_k}|}{|\alpha_{\xi_k}|} \overset{k\to\infty}{\longrightarrow} 0,$$

a contradiction.

We will be done once we show that $(b_{d-2,\xi}), \cdots, (b_{1,\xi}), (b_{0,\xi})$ all belong to $s'(\mathbb{Z}^n)$ by an inductive argument. Suppose that k is the least index such that $(b_{k,\xi}) \notin s'(\mathbb{Z}^n)$. But then by a similar argument as above, it follows from

$$\alpha_\xi \cdot b_{k+1,\xi} + b_{k,\xi} = c_{k+1,\xi}$$

that $(b_{k+1,\xi}) \notin s'(\mathbb{Z}^n)$ (since we have already established that $(\alpha_\xi) \in s'(\mathbb{Z}^n)$). Proceeding in this manner, we eventually obtain that $b_{d-2,\xi} \notin s'(\mathbb{Z}^n)$, which clearly contradicts the last equation in the above equation array, namely that $\alpha_\xi + b_{d-2,\xi} = c_{d-1,\xi}$.

SURJECTIVITY FOR PARTIAL DIFFERENTIAL OPERATORS ON SPACES OF REAL ANALYTIC FUNCTIONS

In this paper, we continue the study of the basic question when

$P(D): A(\Omega) \to A(\Omega)$ is surjective. (1) (1.1)

Here *P(D)* is a partial differential operator with constant coefficients, $\Omega \subset \mathbb{R}^n$ is an open set and $A(\Omega)$ is the space of real analytic functions 011 Ω.

Solutions to this problem have been given mainly by two methods: Hormander (1973) has characterized (1) for *convex* open sets Ω by means of a Phragmen-Lindelof condition valid on the complex variety of P_m. His method has been adapted by several authors for further studies on this problem.

Homiander s criterion is restricted to convex sets Ω by the use of Fourier theory. On the other hand, Kawai (1972) used so-called "good elementary solutions" for *P(D)* to prove (1) for locally hyperbolic operators on special, not necessarily convex bounded open sets Ω for the case of unbounded open sets and further results in the spirit of Kawai's work, and Andersson (1974) for $\Omega = \mathbb{R}^n$).

In Langenbruch (2004) we recently clarified the role of fundamental solutions for our problem, and we gave several characterizations of (1) by means of elementary solutions and also by conditions of type.

In the present paper, a quantitative version of the latter characterization will be proved. Using this new condition and a result of Homiander (1973), we will show that *P(D)* is surjective on $A(\mathbb{R}^n)$

if *P* (*D*) is siuj ective on $A(\Omega)$ for some $\Omega \neq \emptyset$. (2) (1.2)

For convex Ω, this is one of the main results of Homiander (1973). while the question had been open for general Ω. Thus, surjectivity of *P(D)* on $A(\mathbb{R}^n)$ is a general necessary condition for (1).

We also show that surjectivity of partial differential operators on real analytic functions is inherited similarly as for operators on C^∞-functions. In fact, if *P(D)* is surjective on $A(\Omega_j)$ for any $j \in J$ then *P(D)* is surjective on $A(\Omega)$ for

$$\Omega := \Big(\bigcap_{j \in J} \Omega_j\Big)^\circ$$

and

$$\Omega := \liminf_j {}^\circ \Omega_j$$

$:= \{\xi \in \mathbb{R}^n \mid \exists \varepsilon > 0\colon B_\varepsilon(\xi) \subset \Omega_j$ for any but finitely many j }.

Also, if *P(D)* is surjective on $A(\Omega)$, then *P(D)* is surjective 011 $A(\Omega_\varepsilon)$ for any $\varepsilon > 0$

where

$$\Omega_\varepsilon := \{\xi \in \Omega \mid \operatorname{dist}(\xi, \partial\Omega) > \varepsilon\}.$$

Next, we obtain the following result of Andreotti and Nacinovich (1980) and Zampieri (1984): For convex Ω, (1) holds if and only if P(D) is surjective on *A(H)* for any tangent halfspace *H* of Ω.

We finally show an extension of this result for homogeneous operators *P(D)* and open sets Ω with C^1-boundary: In this case. *P(D)* is surjective on A(conv(Ω)) and on *A(H)* for any tangent halfspace of Ω if (1) holds.

A TOPOLOGICAL PALEY-WIENER THEOREM

First we need a topological Paley-Wiener Theorem and in order to do so, we have to topologize the space $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$. For this, introduce the space $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))$ to be the space consisting of holomorphic maps $\psi : \mathfrak{a}^*_{\mathbb{C}} \longrightarrow L^2(B)$ satisfying

$$\|\psi(\lambda)\|_{L^2(B)} \leq C_N e^{R|\operatorname{Im}\lambda|}(1+|\lambda|)^{-N}$$

for all N. We define $|||\psi|||_N$ to be the smallest such constant C_N, and we topologize $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))$ by this family of seminorms. This turns it into a Frechet space. The Weyl invariance still makes sense in this generalized setting, and thus we define $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))_W$ to be the subset of Weyl invariant elements. This is a closed subspace and hence a Frechet space.

We have an obvious inclusion $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W \longrightarrow \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))_W$ and this inclusion turns out to be surjective:

Lemma 2. *It holds that* $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))_W = \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ *as vector spaces.*

Proof. For $\psi \in \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))_W$ define

$$f(x) := \int_{\mathfrak{a}^* \times B} \psi(\lambda, b) e^{(i\lambda+\rho)A(x,b)} |c(\lambda)|^{-2} db\, d\lambda$$

$$= \int_{\mathfrak{a}^*} \langle \psi(\lambda), e^{(-i\lambda+\rho)A(x,\cdot)} \rangle_{L^2(B)} |c(\lambda)|^{-2} d\lambda.$$

Obviously, f is a smooth function. By examining the proof of bijectivity of $\mathcal{F} : \mathscr{D}(X) \longrightarrow \mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$, it is seen that f is supported in the closed R-ball $B_R(eK)$ and that $\widetilde{f} - \psi = 0$ almost everywhere, and thus f is a smooth representative of φ.

Furthermore $\widetilde{f}$ satisfies the stronger growth condition and hence $f \in \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$.

Now $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ inherits the topology from $\mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}}, L^2(B))_W$ (given by the seminorms $|||\cdot|||_N$), and hence it becomes a Frechet space. Furthermore we define $\mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W := \bigcup_{R \in \mathbb{Z}_{>0}} \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ and give it the inductive limit topology.

Theorem 2 (Topological Paley-Wiener). *The Fourier transform* $\mathcal{F} : \mathscr{D}(X) \longrightarrow \mathcal{H}(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ *is a linear homeomorphism. Furthermore* $\widetilde{f} \in \mathcal{H}^R(\mathfrak{a}^*_{\mathbb{C}} \times B)_W$ *if and only if* $f \in \mathscr{D}_R(X)$

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Proof The bijectivity of $\mathcal{F}$ as well as the last claim is stated and proved.

Now we consider $\mathcal{F} : \mathscr{D}_R(X) \longrightarrow \mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ for a given R. For $f \in \mathscr{D}_R(X)$ it is straightforward to check the inequality for each N:

$$\int_{B_R(eK)} |Df(x)||e^{(-i\lambda+\rho)A(x,b)}|dx$$

$$e^{-R|\operatorname{Im}\lambda|}|e^{(-i\lambda+\rho)A(x,b)}| = e^{(\operatorname{Im}\lambda+\rho)A(x,b)-R|\operatorname{Im}\lambda|} \leq e^{\rho(A(x,b))} \leq e^{R|\rho|}.$$

$$|||\mathcal{F}f|||_N \leq C \sup_{\lambda\in\mathfrak{a}_{\mathbb{C}}^*, b\in B} e^{-R|\operatorname{Im}\lambda|}$$

where D is the invariant differential operator (of order $2N$) on X corresponding to the invariant polynomial and where C is a constant depending on N and R. Since $x \in \overline{B_R(eK)}$ and hence we see that

Hence we get $|||\mathcal{F}f|||_N \leq C\|f\|_{2N}$, where $\|\cdot\|_{2N}$ is one of the standard semi norms on $\mathscr{D}_R(X)$, i.e. $\mathcal{F} : \mathscr{D}_R(X) \longrightarrow \mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ is continuous.

Thus the Fourier transform is a homeomorphism $\mathscr{D}_R(X) \xrightarrow{\sim} \mathcal{H}^R(\mathfrak{a}_{\mathbb{C}}^* \times B)_W$ since these spaces are Frechet. Hence it is also a homeomorphism when defined on $\mathscr{D}(X)$.

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