GNITED MINDS
Journals

Journal of Advances in Science and Technology

Vol. X, Issue No. XX, November-2015, ISSN 2230-9659

AN ANALYSIS ON HARMONIC FUNCTION FOR MEASURING HYPERGROUPS

AN INTERNATIONALLY INDEXED PEER REVIEWED & REFEREED JOURNAL

www.ignited.in

Journal of Advances in Science and Technology
Vol. X, Issue No. XX, November-2015, ISSN 2230-9659

An Analysis on Harmonic Function for Measuring Hypergroups

Neeta Mahesh Patil

Mathematics, Ph. D. Scholar, AISECT University, Bhopal

Abstract *– We initiate a study of harmonic functions on hypergroups. In particular, we introduce the concept of a nilpotent hypergroup and show such hypergroup admits an invariant measure as well as a Liouville theorem for bounded harmonic functions. Further, positive harmonic functions on nilpotent hypergroups are shown to be integrals of exponential functions. For arbitrary hypergroups, we derive a Harnack inequality for positive harmonic functions and prove a Liouville theorem for compact hypergroups. We discuss an application to harmonic spherical functions.*

The main task in this article is to give the necessary and sufficient conditions guarantees that the product of two positive definite functions defined on a hypergroup X is also positive definite on X. Also, we prove that a continuous function with compact support ψ is negative definite if and only if $exp(-t\psi)$ is positive definite for each t > 0. Moreover, we will give some relations between the class of completely monotonic functions on a hypergroup and the set of $\tau-$ positive functions.

---------------------------♦-----------------------------

INTRODUCTION

The modern approach to harmonic analysis on a Lie group treats the representations of the group as the central objects of study, while characters are treated as important but auxiliary objects associated to representations. This is in direct contrast to the modern approach to harmonic analysis on a finite group, which treats the determination and study of the characters of the group as the primary problem, and considers representations as important but auxiliary objects associated to characters.

For finite groups, the reason for this is three fold; 1) the determination of the irreducible characters is a vastly simpler problem than the determination of the irreducible representations 2) almost all of the standard problems of harmonic analysis may be answered solely by means of the character theory and 3) historically, the theory of characters has preceded that of the theory of representations.

This suggests the following interesting question - might it be possible to develop harmonic analysis on a Lie group as essentially a theory of characters, and thereby finesse the present difficulties and technicalities in modern representation theory? [This is not an entirely new idea- in fact Harish Chandra's pivotal work on the existence of discrete series for non-compact semi-simple groups takes this view.]

In this paper we show how such an approach may be initiated, and applied to various classes of groups. The first problem is of course - how does one define a character of a group if one does not know what a representation is? Our answer to this question is a variant of the one Ftobenius would have given- the characters of a finite group, say, are exactly those functions on the set of conjugacy ·classes of the group which respect the natural algebraic structure of this set. This algebraic structure has a probabilistic nature and is called an abelian hypergroup. We therefore replace the problem of non-commutative harmonic analysis on the group G with the problem of abelian harmonic analysis on the hypergroup of conjugacy classes, which we call the class hypergroup of G. This allows us to sketch a straight forward algorithm for the construction of the character table of any given finite group.

It also provides a general framework for the study of harmonic analysis on an arbitrary group. When G is a Lie group, we show that one may expect an intimate relationship between the class hypergroup of G and the hypergroup of adjoint orbits. This provides an explanation for the general effectiveness of Kirillov theory in harmonic analysis since, formally at least, the dual object of the hypergroup of adjoint orbits is the hypergroup of co-adjoint orbits, although we also see that in general Kirillov theory will provide at best a 'linear approximation' to the unitary dual of G on account of the global difference between the class hypergroup and the hypergroup of adjoint orbits.

We therefore propose a program for the determination and study of the unitary dual of a Lie group G which both incorporates and extends Kirillov theory.

It is well known that Riemannian symmetric spaces can be represented as homogeneous spaces G/K of Lie groups G. Recently, harmonic functions on Riemannian symmetric spaces have been studied in C-H. Chu, A.T-M. Lau (2010) via convolution semigroups of measures on G, where the harmonic spherical functions identify as functions on the double coset space G//K. Since double coset spaces are special examples of hypergroups on which the Borel measures have a convolution structure, it is natural to consider harmonic functions on the wider class of hypergroups where convolution can be exploited. Our objective is to develop a basic theory of harmonic functions on hypergroups which is applicable to a large class of examples besides symmetric spaces, for instance, the group orbit spaces, spaces of conjugacy classes and dual spaces of compact groups.

Given a Borel measure σ on a hypergroup H, a Borel function f on H is called σ - harmonic if it satisfies the convolution equation

$$f = f * \sigma.$$

We first introduce the concept of a nilpotent hypergroup and focus our attention on them. Commutative hypergroups are nilpotent and include, for example, double coset spaces G//K of Gelfand pairs (G, K), polynomial hypergroups and. hypergroups arising from free groups, discrete semigroups, quantum groups and various differential equations. We show that a nilpotent hypergroup admits an invariant measure and the Liouville theorem holds for these hypergroups. Further, we show that positive harmonic functions on metrisable nilpotent hypergroups are integrals of exponential functions.

For arbitrary hypergroups with an invariant measure, we derive a Harnack inequality for positive harmonic functions and prove a Liouville theorem for compact hypergroups. All the above results should translate into interesting applications to concrete examples of hypergroups. As an example of such application, consider the double coset space G//K of a Gelfand pair (G, K). It is a commutative hypergroup and therefore Liouville theorem holds. It follows that, given an adapted radial probability measure σ on G. the bounded σ-harmonic spherical functions on G must be constant. In the case of $G = SU(1,1)$ and K the subgroup of rotations, this crucial fact was the key in Y. Benyamini et al. (2002) to showing that the bounded σ-harmonic functions on the unit disc $SU(1,1)/K$ are exactly the bounded harmonic functions of the Laplacian. In fact, the latter result should hold for more general Lie groups other than $SU(1,1)$, as noted in Y. Benyamini et al. (2002),

POSITIVE HARMONIC FUNCTIONS ON HYPERGROUPS

In this section, we derive a Harnack inequality for positive harmonic functions on a hypergroup G and, if G is nilpotent and metrisable, we determine these functions completely.

To describe positive, possibly unbounded, σ-harmonic functions on a hypergroup G, one can make use of Choquet's representation theory on cones of Radon measures in $M_+(G)$. Let $\mathcal{C}$ be

a subcone of $M_+(G)$. A measure $\mu \in \mathcal{C}$ is called extremal in $\mathcal{C}$ if every $\nu \in \mathcal{C}$ with $\nu \leqslant \mu$ is a positive multiple of μ. Let $\partial\mathcal{C}$ be the set of extremal measures in $\mathcal{C}$.

For a Radon measure $\nu \in M(G)$ and a real Borel function / on G, we define a measure $f\nu \in M(G)$ by $f\nu(E) = \int_E f\,d\nu$ for each Borel set $E \subset G$ If G admits a right Haar measure ω and if / is a positive solution of the equation $f * \sigma = f$ then, as in, the measure $\mu = f\omega \in M_+(G)$ satisfies the equation

$$\mu * \sigma = (f\omega) * \sigma = (f * \sigma)\omega = f\omega = \mu.$$

Therefore one can determine all positive solutions of $f * \sigma = f$ once one describes every element in the cone

$$\mathcal{H}_\sigma = \{\mu \in M_+(G) : \mu * \sigma = \mu\}$$

To achieve this, we consider the larger cone

$$\mathcal{C}_\sigma = \{\mu \in M_+(G) : \mu * \sigma \leqslant \mu\}$$

w hich is closed in the weak* topology $\sigma(M(G), C_c(G))$. Hence, if G is separable and metrisable, then $\mathcal{C}_\sigma$ is weak* complete and $\partial\mathcal{C}_\sigma$ is a Borel set. By Choquet's representation theory, each $\mu \in \mathcal{C}_\sigma$ has an integral representation

$$\mu = \int_{\partial\mathcal{C}_\sigma} \nu\, d\mathcal{P}(\nu)$$

where, for $\mu \in \mathcal{H}_\sigma \subset \mathcal{C}_\sigma$ the measure $\mathcal{P}$ is supported on the Borel set $\partial\mathcal{H}_\sigma = \mathcal{H}_\sigma \cap \partial\mathcal{C}_\sigma$. If G is nilpotent, we describe $\partial\mathcal{H}_\sigma$ and hence $\mathcal{H}_\sigma$, completely.

Lemma. Let cr be a positive non-degenerate Radon measure on a hypergroup G. Let f be a non-negative continuous a-harmonic function on G. Then either f is identically zero or f is strictly positive everywhere on G.

Proof. Let / be nonzero somewhere. We show $f(e) > 0$.

Suppose otherwise, we deduce a contradiction. The open set

$$V = \{x \in G: \ f(\bar{x}) > 0\}$$

is nonempty and hence $\sigma^n(V) > 0$ for some $n \in \mathbb{N}$ by non-degeneracy of σ. Hence

$$0 = f(e) = \int_G f(\bar{x})\, d\sigma^n(x) \geqslant \int_V f(\bar{x})\, d\sigma^n(x) > 0$$

which is impossible. This proves $f(e) > 0$

For each $z \in G$, the left translate $f(z * \cdot)$ is not identically zero since $f(x) > 0$ for some $x \in G$ implies $f > 0$ on some open neighbourhood V of jc and $f(z * \bar{z} * x) \geqslant \int_V f\, d(\delta_z * \delta_{\bar{z}} * \delta_x) > 0$ where $x \in \operatorname{supp}(\delta_z * \delta_{\bar{z}} * \delta_x)$ Repeating the previous arguments to the σ-harmonic function $f(z * \cdot)$ gives $f(z) = f(z * e) > 0.$

Let G be a hypergroup which has a right Haar measure ω. Denote by Δ the modular function of G. Let $\sigma = \varphi\omega \in M_+(G)$ be an absolutely continuous probability measure on G, with compact support. By taking an infinite series involving repeated convolutions of σ with itself, one can find a strictly positive lower semicontinuous function ψ on G such that $\mu \in \mathcal{H}_\sigma$ implies $\mu * \sigma_1 = \mu$ and $\mu * \psi = \mu * \varphi$.

where $\sigma_1 = \psi\omega$ is a probability measure on G.

We note that

$$\mu * \psi(e) = \int_G \psi(\bar{x})\Delta(\bar{x})\, d\mu(x)$$

Theorem. Let G be a hypergroup which admits a right Haar measure co and let σ be an ω-absolutely continuous non-degenerate probability' measure on G, with compact support. Then for each compact subset K of G, there is a constant $c_K > 0$ such that

$$f(x * y) \leqslant c_K f(x) \quad (x \in G,\ y \in K)$$

for all positive continuous σ-harmonic functions f on G.

Proof. For each $x \in G$ and each positive continuous σ-harmonic function / on G, the function $g: G \to (0, \infty)$ defined by

$$g(y) = \frac{f(x * y)}{f(x)} \quad (y \in G)$$

is positive continuous σ-harmonic and its value at e is 1. therefore we have, by the above remark, $\frac{f(x*y)}{f(x)} \leqslant c_K$ for all $y \in K$.

POSITIVE AND NEGATIVE DEFINITE FUNCTIONS ON HYPERGROUPS

Positive definite functions –

A hypergroup (X,*) is called commutative if (M(X),+,*) is a commutative algebra, and hermitian if the involution $\bar{\ }$ is the identity map. Its easy to prove that every her- mitian hypergroup is commutative. A locally bounded measurable function $\chi : X \to \mathbb{C}$ is called a semicharaeter if $\chi(e) = 1$ and $\chi(x * y^-) = \chi(x)\overline{\chi(y)}$ for all $x, y \in X$. Every bounded semicharacter is called a character. If the character is not locally null then it must be continuous. The dual X^* of X is just the set of continuous characters with the compact-open topology in which case X^* must be locally compact. In this paper we will be concerned with continuous characters on hypergroups. A locally bounded measurable function $\phi : X \to \mathbb{C}$ is said to be positive definite if

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i\overline{c_j}\phi(x_i * x_j^-) \geq 0$$

for all choice of $x_1, x_2, ..., x_n \in X, c_1, c_2, ..., c_n \in \mathbb{C}$ and $n \in \mathbb{N}$. The following two lemmas are in fact, an adaption of whatever done for semigroups. We will not repeat the proof, wherever the proof for semigroups can be applied to the hypergroups with necessary modification.

Negative definite functions –

One should be observe that, a function ψ is negative definite if and only if $exp(-t\psi)$ is positive definite for each t > 0. While this result holds for all semigroups it is not clear how to prove the 'only if' part for hypergroups since the usual technique do not apply(the 'if' part always holds provided that $\mathrm{Re}\psi$ is locally lower bounded). The problem is that except when x or y belong to the maximal subgroup of the hypergroup $exp(-t\psi(x*y))$ and $exp(-t\psi)(x*y)$ are usually not equal so that other methods have to be used to overcome this. A locally bounded measurable function q is called a quadratic form if

$$q(x*y)+q(x*y^-)=2q(x)+2q(y)$$

for all $x,y\in X$ and additive if $q(x*y)=2q(x)+2q(y)$ for all $x,y\in X$. In the case X is hermitian, that when X carries the identity involution, then every quadratic form is an additive function and every negative definite function is real. A locally bounded measurable function $\psi:X\to\mathbb{C}$ is said to be negative definite if $\psi(x^-)=\overline{\psi(x)}$ and

$$\sum_{i=1}^{n}\sum_{j=1}^{n}c_i\overline{c_j}\phi(x_i*x_j^-)\le 0$$

for all choice of $x_1,x_2,...,x_n\in X,\ n\in\mathbb{N}$ and all $c_1,c_2,...,c_n\in\mathbb{C}$ that satisfy $\sum_{i=1}^{n}c_i=0$.

A key result in the study of negative definite functions on hypergroups is the following Levy-Khinchin represent at ion

$$\psi(x)=\psi(e)+q(x)+\int_{\hat{X}\backslash\{1\}}(1-Re(\chi(x)))d\eta(\chi)$$

for all $x\in X$ where q is a nonnegative quadratic form on X and $\eta\in M_+(\hat{X}\backslash\{1\})$. Both q and the integral part $\psi(x)-\psi(e)-q(x)$ belong to the set of negative definite function on X and the pair (q,η) is uniquely determined by ψ with q being given by

$$q(x)=lim\{\frac{\psi(x^{*n})}{n^2}+\frac{\psi((x*x)^{*n})}{2n}\}$$

NILPOTENT GROUPS

If G is a connected, simply-connected nilpotent Lie group, then the exponential map is a diffeomorphism from $\mathfrak{g}$ to G. This suggests there should be a strong connection between $\mathcal{C}(G)$ and $\mathcal{C}(\mathfrak{g})$, if these objects really exist; in fact Kirillov theory leads us to predict that $\mathcal{C}(G)\simeq\mathcal{C}(\mathfrak{g})$. The typical conjugacy class is non-compact however, so while it does carry a G-invariant measure, such a measure will not generally be a probability measure. Furthermore the convolution of two such measures may easily not exist, at least in the usual sense. Nevertheless we have the following result.

Theorem. Let $\mathcal{O}_i\subseteq\mathfrak{g}$, i = 1,2,3 be adjoint orbits and $C_i\subseteq G$ the corresponding conjugacy class exp $\mathcal{O}_i=C_i$. Then $\mathcal{O}_3\subseteq\mathcal{O}_1+\mathcal{O}_2$ if and only if $C_3\subseteq C_1C_2$.

Since the proof is pertinent here, we recall the main idea. Working in the free Lie algebra generated by X and Y (and the corresponding algebra of formal power series) and letting X • Y = [X,Y]$_y$ X * Y = In(exp X exp Y) and $X^Y=\exp(Y)\cdot X$, one can show that there exists formal power series A(X, Y) and B(X, Y) such that $X*Y=X^{A(X,Y)}+Y^{B(X,Y)}$.

Furthermore the assignment $Z=X^{A(X,Y)},\ W=Y^{B(X,Y)}$ is invertible in the sense that one can write X and Y as similar formal power series in Z and W. These considerations prove that $\exp(\mathcal{O}_X+\mathcal{O}_Y)=\exp\mathcal{O}_X\exp\mathcal{O}_Y$ but they actually prove more, namely that if we define $A:\mathcal{O}_X\times\mathcal{O}_Y\to\mathfrak{g}$ by $A(X',Y')=X'+Y'$ and by $M(X',Y')=X'*Y'$, then we can find a bijection $\eta:\mathcal{O}_X\times\mathcal{O}_Y\to\mathcal{O}_X\times\mathcal{O}_Y$ such that $M=A\circ\eta$.

This is nothing but a formal proof of our conjecture $\mathcal{C}(G)\simeq\mathcal{C}(\mathfrak{g})$. If G was a finite nilpotent group with a Lie algebra 0 and an exponential map $\exp:\mathfrak{g}\to G$ which was both a bijection and satisfied the Baker-Campbell-Hausdorff formula, then the above argument would allow us immediately to deduce that $\mathcal{C}(G)\simeq\mathcal{C}(\mathfrak{g})$. Are there such groups? Yes there axe - if G is a finite (p — .1) step nilpotent p-group (for some prime p) then Howe has shown that there is an abelian group 0 with the structure of a (p — 1) step nilpotent Lie algebra and a bijection $\exp:\mathfrak{g}\to G$ satisfying the Campbell-Hausdorff formula. We have thus proved.

REFERENCES

- C. Berg, J. P. R. Christensen and P. Ressel, Harmonic analysis on semigroups. Theory of positive definite and related functions, Graduated texts in Math. 100, Springer-Verlag, Berlin-Heidelberg-New-York, (2004).
- C.-H. Chu, C.-W. Leung, The convolution equation of Choquet and Deny on [IN]-groups, Integral Equations Operator Theory 40 (2001) 391–402.
- C-H. Chu, A.T-M. Lau, Harmonic functions on topological groups and symmetric spaces, Math. Z., online March 2010, doi:10.1007/s00209-010-0688-3, in press.
- Dunkl, C.F., The measure algebra of a locally compact bypergroup, Trans. Amer. Math. Soc. 179 (2003) 331-348.
- Howe, R., On representations of discrete, finitely-generated, torsion free nilpotent

groups, Pacific J. of Math. 73 (2) (2007) 281-305.

- Jewitt, R.L, Spaces with an abstract convolution of measures, Advances in Math, 18 (2005) 1-101.

- R. Bloom, H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups,Walter de Gruyter, Berlin, 2005.

- R. Spector, Apercu de la theorie des hypergroups in analyse harmonique sur les groups de Lie, Lecture Notes in Math. 497, Springer Verlag, New York, (2005) .

- Spector, R., Measures invariants sur les bypergroups, Trans. Amer. Math. Soc. 239 (2008) 147-165.

- Varadarajan, V., Harmonic Analysis on Real Reductive Groups, LNM 576 SpringerVerlag, Berlin 2007.

- W. R. Bloom and H. Heyer, Harmonic analysis of probability measures on hypergroups, de Gruyter, Berlin, (2005).

- Wildberger, N.J., On a relationship between adjoint orbits and conjugacy classes of a Lie group, Canad. Math. Bull. 33 (3) (2000) 297-304.

- Y. Benyamini, Y.Weit, Harmonic analysis on spherical functions on SU(1.1), Ann. Inst. Fourier 42 (2002) 671–694.

- Y. Weit, On Furstenberg’s characterization of harmonic functions on symmetric spaces, Israel J. Math. 114 (1999) 265–269.