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Nonlinear Dispersive Equations on Modulation Spaces by NLS Approximations

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Abstract – The aim of this paper is to propose and analyze various numerical methods for some representative classes of nonlinear dispersive equations, which mainly arise in the problems of quantum mechanics and nonlinear optics. Extensive numerical results are also reported, which are geared towards demonstrating the efficiency and accuracy of the methods, as well as illustrating the numerical analysis and applications. By using tools of time-frequency analysis, we obtain some improved local well-posedness results for the NLS, equations with data in modulation spaces. This is the first time in which particle methods are being used for solving such equations. We numerically test our new method for a variety of linear and nonlinear problems. In particular we are interested in nonlinear equations which generate structures that have non-smooth fronts. It is remarkable to see that our particle method is capable of capturing the nonlinear regime of a compacton-cotnpacton type interaction.

Keywords: Nonlinear, Dispersive, Equations, Modulation, Spaces, NLS, Approximations, Method, Problems, Numerically, Equations, etc.

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INTRODUCTION

The goal of the present study is to continue our explorations of the effect of periodicity on rough initial data for nonlinear evolution equations in the context of two important examples: the nonlinear Schrodinger (nlS) and Korteweg-deVries (KdV) equations, possessing, respectively, elementary second and third order monomial dispersion. Our basic numerical tool is the operator splitting method, which serves to highlight the interplay between the behaviors induced by the linear and nonlinear parts of the equation. Earlier rigorous results concerning the operator splitting method for the Korteweg-deVries, generalized Korteweg- deVries, and nonlinear Schrodinger equations can be found in other studies (Cottet 2000). We also refer the reader and the references therein for a discussion of alternative numerical schemes and convergence thereof for L^2 initial data on the real line.

Nonlinear Schrodinger (NLS) for $\chi^{(3)}$ medium. A comparison between Maxwell solutions and those of an extended NLS also showed that the cubic NLS approximation works reasonably well on short stable ID pulses (Kaya 2005). Mathematical analysis on the validity of NLS approximation of pulses and counter-propagating pulses of ID sine-Gordon equation has been carried out. However, in 2D, the envelope approximation with the cubic focusing NLS breaks down, because critical collapse of the cubic focusing NLS occurs in finite time (and references therein).

REVIEW OF LITERATURE:

The propagation and interaction of spatially localized optical pulses (so-called *light bullets* (LBs)) with particle features in several space dimensions are of both physical and mathematical interests. They have been found useful as information carriers in communication, as energy sources, switches and logic gates in optical devices. Such LBs have been observed in numerical simulations of the full Maxwell system with instantaneous Kerr ($\chi^{(3)}$or cubic) nonlinearity in 2D. They are short femtosecond pulses that propagate without essentially changing shapes over a long distance and have only a few EM (electromagnetic) oscillations under their envelopes.

In ID, the Maxwell system modeling light propagation in nonlinear media admits constant-speed traveling waves as exact solutions, also known as the light bubbles (unipolar pulses or solitons) (D'Aprile, 2004). In several space dimensions, constant-speed traveling waves (mono-scale solutions) are harder to come by. Instead, space-time oscillating (multiple-scale) solutions are more robust. The so-called LBs are of multiple- scale structures with distinct phase/group velocities and amplitude dynamics. Even though direct numerical simulations of the full Maxwell system are motivating, asymptotic approximation is necessary for analysis in several space dimensions. The approximation of ID Maxwell system has been extensively studied. Long pulses are well approximated via envelope approximation by the cubic focusing nonlinear Schrodinger (NLS) for $\chi^{(3)}$ medium. A comparison between Maxwell solutions and those of an extended NLS also showed that the cubic NLS approximation works reasonably well on short stable ID pulses. Mathematical analysis on the validity of NLS approximation of pulses and counter-propagating pulses of ID sine-Gordon equation has been carried out. However, in 2D, the envelope approximation with the cubic focusing NLS breaks down, because critical collapse of the cubic focusing NLS occurs in finite time (and references therein) (Frutos 2005). On the other hand, due to the intrinsic physical mechanism or material response, Maxwell system itself typically behaves fine beyond the cubic NLS collapse time. One example is the semi-classical two level dissipation less Maxwell-Bloch system where smooth solutions persist forever. It is thus a very interesting question how to modify the cubic NLS approximation to capture the correct physics for modeling the propagation and interaction of light signals in 2D Maxwell type systems. One approach will be discussed in the following.

Considering the transverse electric regime, after taking a distinguished asymptotic limit of the two level dissipation less Maxwell-Bloch system studied in, Xin found that the well-known sine-Gordon (SG) equation (1)—(2) also admits 2D LBs solutions (Degond 2000). In the SG equation (3)—(4), it is well-known that the energy is conserved.

$$E^{\mathrm{SG}}(t) := \int_{\mathbb{R}^2} \left[(\partial_t u)^2 + c^2 |\nabla u|^2 + 2G(u)\right] \mathrm{d}x, \quad t \geq 0, \tag{1}$$

With

$$G(u) = \int_0^u \sin(s)\mathrm{d}s = 1 - \cos(u), \tag{2}$$

Direct numerical simulations of the SG equation in 2D were performed in, which are much simpler tasks than simulating the full Maxwell system. Moving pulse solutions being able to keep the overall profile over a long time were observed, just like those in Maxwell system. See also for related breather-type solutions of the SG equation in 2D based on a modulation analysis in the Lagrangian formulation.

Also, with the SG-LBs as starting point one can look for a modulated planar pulse solution of the SG equation in the form:

$$u(\mathbf{x},t) = \varepsilon A(\varepsilon(x-\nu t), \varepsilon y, \varepsilon^2 t) e^{i(kx-\omega(k)t)} + \text{c.c.} + \varepsilon^3 u_2, \quad \mathbf{x} = (x,y) \in \mathbb{R}^2,\ t \geq 0, \tag{3}$$

Where $0 < \varepsilon \ll 1,\ \omega = \omega(k) = \sqrt{1+c^2k^2},\ \nu = \omega'(k) = c^2k/\omega$, the group velocity, and c.c. refers to the complex conjugate of the previous term. Plugging, setting $X = \varepsilon(x-\nu t), Y = \varepsilon y$ and $T = \varepsilon^2 t$, calculating derivatives, expressing the sine function in series and removing all the resonance terms, one can obtain the following complete perturbed NLS equation:

$$-2i\omega\partial_T A + \varepsilon^2 \partial_{TT} A = \frac{c^2}{\omega^2}\partial_{XX} A + c^2 \partial_{YY} A + 2\varepsilon\nu\partial_{XT} A + |A|^2 A \sum_{l=0}^{\infty} \frac{(-1)^l (\varepsilon|A|)^{2l}}{(l+1)!(l+2)!}, \quad T > 0, \tag{4}$$

Where $A := A(\mathbf{X},T),\ \mathbf{X} = (X,Y) \in \mathbb{R}^2$, is a complex-valued function. This new equation is second order in space-time and contains a non praxialist term, a mixed derivative term, and a novel nonlinear term which is saturating for large amplitude.

Introducing the scaling variables $\tilde{X} = (\omega/c)X,\ \tilde{Y} = Y/c$ and $\tilde{T} = T/(2\omega)$, substituting and then removing all˜, one gets a standard perturbed NLS equation,

$$i\partial_T A - \frac{\varepsilon^2}{4\omega^2}\partial_{TT} A = -\Delta A - \frac{\varepsilon c k}{\omega}\partial_{XT} A + f_\varepsilon(|A|^2)A, \qquad T > 0,$$

With initial conditions,

$$A(\mathbf{X},0) = A^{(0)}(\mathbf{X}), \quad \partial_T A(\mathbf{X},0) = A^{(1)}(\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^2,$$

Where,

$$\rho = |A|^2, \quad f_\varepsilon(\rho) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}\varepsilon^{2l}\rho^{l+1}}{(l+1)!(l+2)!}. \tag{5}$$

In fact, equation can be viewed as a perturbed cubic NLS equation with both a saturating nonlinearity (series) term and no paraxial terms (the A_{TT} and $A_X t$ terms). It conserves the *energy*, i.e.,

$$E^{\mathrm{PNLS}}(T) := \int_{\mathbb{R}^2} \left[\frac{\varepsilon^2}{4\omega^2}|A_T|^2 + |\nabla A|^2 + F_\varepsilon\left(|A|^2\right)\right] \mathrm{d}\mathbf{X} \equiv E^{\mathrm{PNLS}}(0),\ T \geq 0,$$

With

$$F_\varepsilon(\rho) = \int_0^\rho f_\varepsilon(s)\,\mathrm{d}s = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}\varepsilon^{2l}\rho^{l+2}}{(l+1)!(l+2)!(l+2)},$$

And has the *mass* balance identity

$$\frac{\mathrm{d}}{\mathrm{d}T}\left(\int_{\mathbb{R}^2} |A|^2\,\mathrm{d}\mathbf{X} - \frac{\varepsilon^2}{2\omega^2}\mathrm{Im}\int_{\mathbb{R}^2} A_T A^*\,\mathrm{d}\mathbf{X}\right) = \frac{2\varepsilon\nu}{c}\mathrm{Im}\int A_X A_T^*\,\mathrm{d}\mathbf{X}.$$

In addition, the perturbed NLS equation is globally well-posed and does not have finite-time collapse, i.e., for any given initial data $A^{(0)}(\mathbf{X}) \in H^2(\mathbb{R}^2)$ and $A^{(1)}(\mathbf{X}) \in H^1(\mathbb{R}^2)$, the initial value problem of (1.25) with initial conditions (5.6) has a unique global solution $A \in C([0,\infty]; H^2(\mathbb{R}^2)), A_T \in C([0,\infty]; H^1(\mathbb{R}^2))$, and $A_{TT} \in C([0,\infty]; L^2(\mathbb{R}^2))$

In practice, the infinite series of nonlinearity could be truncated to finite terms with focusing-defocusing cycles. Denote

$$f_\varepsilon^N(\rho) = \sum_{l=0}^{N} \frac{\varepsilon^{4l}\rho^{2l+1}}{(2l+1)!(2l+2)!}\left[-1+\frac{\varepsilon^2\rho}{(2l+2)(2l+3)}\right], \quad (1.31)$$

Then the perturbed NLS equation can be approximated by the following truncated NLS equation:

$$i\partial_T A - \frac{\varepsilon^2}{4\omega^2}\partial_{TT}A = -\Delta A - \frac{\varepsilon ck}{\omega}\partial_{XT}A + f_\varepsilon^N(|A|^2)A, \qquad T > 0. \quad (1.32)$$

Similar to the proof for the perturbed NLS equation, one can show that the truncated NLS equation with the initial conditions also conserves the *energy*, i.e.,

$$E_N^{\text{PNLS}}(T) := \int_{\mathbb{R}^2}\left[\frac{\varepsilon^2}{4\omega^2}|A_T|^2 + |\nabla A|^2 + F_\varepsilon^N\left(|A|^2\right)\right] d\mathbf{X} \equiv E_N^{\text{PNLS}}(0),\ T \geq 0, \quad (1.33)$$

With

$$F_\varepsilon^N(\rho) = \int_0^\rho f_\varepsilon^N(s)\,ds = \sum_{l=0}^{N}\frac{\varepsilon^{4l}\rho^{2l+2}}{(2l+1)!(2l+2)!(2l+2)}\left[-1+\frac{\varepsilon^2\rho}{(2l+3)^2}\right], \quad (1.34)$$

And has the *mass* balance identity.

When $\varepsilon = 0$, the perturbed NLS equation and its approximation collapse to the well-known critical cubic focusing NLS equation:

$$i\partial_T A = -\Delta A - \frac{1}{2}|A|^2 A, \quad T > 0, \quad (1.35)$$

With initial condition,

$$A(\mathbf{X}, 0) = A^{(0)}(\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^2. \quad (1.36)$$

It is well-known that this cubic NLS equation conserves the energy, i.e.,

$$E^{\text{CNLS}}(T) := \int_{\mathbb{R}^2}\left[|\nabla A|^2 - \frac{1}{4}|A|^4\right] d\mathbf{X} \equiv \int_{\mathbb{R}^2}\left[|\nabla A^{(0)}|^2 - \frac{1}{4}|A^{(0)}|^4\right] d\mathbf{X}, \quad (1.37)$$

And collapses in finite-time when the initial energy $E^{\text{CNLS}}(0) < 0$, which motivates different choices of initial data for numerical experiments.

CONCLUSION:

The main goal of this study is to construct an approximate analytical solution for nonlinear dispersive equations. We have achieved this goal by applying reduced differential transform method. Two special cases are chosen to illustrate the effectiveness and efficiency of the method. Results are compared with analytical solutions. The main advantage of the NLS is to provide the user an analytical approximation to the solution, in many cases, an exact solution, in a rapidly convergent sequence with elegantly computed terms. NLS needs small size of computation contrary to other numerical methods, converges rapidly and introduces a significant improvement solving nonlinear dispersive equations over existing methods.

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