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A Study on Cubature Formulas and ISO Metric Embeddings

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Abstract – The subject of the present work arose in a connection with well-known and deeply developed problem about almost Euclidean subspaces of normed spaces. This is far from being a complete list of the publications about the subject. As a rule, a normed space does not contain Euclidean subspaces of dimensions greater than one. However, the famous Dvoretzky theorem states the existence of almost Euclidean subspaces of all normed spaces of sufficiently big dimensions.

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INTRODUCTION

The presence of an Euclidean subspace of a dimension m>=2, an Euclidean plane does exist as well. In the latter case, the unit sphere of the given space contains a circle. The spheres of such a kind arise in a natural way rarely. For example, one can prove that the real space may contain an Euclidean subspace in the only case of even p while there is an example of 2-dimensional Euclidean plane in the real space. An example of Euclidean plane in the real space was presented. A proof of existence of m-dimensional Euclidean subspace of the real space for sufficiently being *n* depending on m and p, n >=NR (m,p), was outlined in the same work . Such an approach also yields an upper bound for NR(m,p). Note that the Euclidean subspaces in lnp are just the images of isometric embeddings lm2 -> lnp. Later on we prefer to speak about the embeddings.

EXAMPLE 1: The identity

$$\xi_1^4 + \left(\frac{\xi_1+\xi_2\sqrt{3}}{2}\right)^4 + \left(\frac{\xi_1-\xi_2\sqrt{3}}{2}\right)^4 = (\xi_1^2+\xi_2^2)^2$$

Shows that mapping

$$\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix} \mapsto \begin{bmatrix}\xi_1\\ (\xi_1+\xi_2\sqrt{3})/2 \\ (\xi_1-\xi_2\sqrt{3})/2\end{bmatrix}$$

Is an isometric embedding l22 -> l 34.

EXAMPLE 2: The Lucas Identity

$$\sum_{1\le i<k\le 4}(\xi_i+\xi_k)^4 + \sum_{1\le i<k\le 4}(\xi_i-\xi_k)^4 = 6(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)^2$$

Defines an embedding l42 -> l 124.

In such a way , one can to interpret a whole series of another classical identities.

EXAMPLE 3: The Identity

$$\frac{1}{n}\sum_{k=1}^{n}\left(\xi_1\cos\frac{\pi k}{n}+\xi_2\sin\frac{\pi k}{n}\right)^p = \frac{p!}{2^p(\frac{p}{2})!^2}(\xi_1^2+\xi_2^2)^{\frac{p}{2}}, \quad n=\frac{p}{2}+1,$$

Defines an isometric embedding. Moreover, in this case n is the minimal possible for given p, so that NR(2,p) = (p/2+1)

In the independent works , an equivalence between isometric embeddings of real spaces lm2 -> l n cubature formulas on the unit sphere Sm-1 C lm2 was established and some lower bounds for NR (m,p) were obtained on this base. In addition, a group orbits method for constructing of isometric embeddingswas developed. For cubature formulas such a method comes back to Ditkin and Ljusternik and Sobolev and was widely applied in order to construct cubature formulas equal weights. The concept of spherical designs was introduced, the paper of Delsarte, Goethals and Seidel containing a series of important examples and fundamentals bounds. The problem of existence of spherical designs was in general open. Some further constructions were done. The theory of general cubature formulas was initiated by Radon and continued by Stroud and Mysovskikh. Now it is developed subject.

LITERATURE REVIEW

Jacobi Polynomials

Let us start with a preliminary information . First of all, we recall that the classical Jacobi Polynomial is the kth member of the sequence of polynomials which are orthogonal on [-1,1] with respect to the Jacobi weight

$$\omega_{\alpha,\beta}(u) = (1-u)^{\alpha}(1+u)^{\beta} \quad (\alpha,\beta > -1)$$

Or, equivalently, to the normalized Jacobi weight

$$\Omega_{\alpha,\beta}(u) = \frac{\omega_{\alpha,\beta}(u)}{\tau_{\alpha,\beta}}; \quad \tau_{\alpha,\beta} = \int_{-1}^{1} \omega_{\alpha,\beta}(u)\,du$$

An explicit expression for Jacobi Polynomial is

$$P_k^{(\alpha,\beta)}(u) = \frac{1}{2^k}\sum_{\nu=0}^{k}\binom{\alpha+k}{\nu}\binom{\beta+k}{k-\nu}(u-1)^{k-\nu}(u+1)^{\nu},$$

Obviously,

$$\deg P_k^{(\alpha,\beta)} = k, \quad P_k^{(\alpha,\beta)}(1) = \binom{\alpha+k}{k}$$

And

$$P_k^{(\beta,\alpha)}(-u) = (-1)^k P_k^{(\alpha,\beta)}(u).$$

In particular, the polynomials are even for even k and for odd for odd k. The latter polynomials are in the essence the Gegenbauer polynomials. More precisely, the Gegenbauer polynomial is defined as

$$C_k^{\nu}(u) = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(2\nu+k)}{\Gamma(2\nu)\Gamma(\nu+k+\frac{1}{2})}P_k^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(u), \quad \nu > -\frac{1}{2},$$

So that

$$\deg C_k^{\nu} = k, \quad C_k^{\nu}(1) = \binom{2\nu+k-1}{k}.$$

In addition,

$$C_k^{\nu}(-1) = (-1)^k\binom{2\nu+k-1}{k}.$$

With a fixed v the Gegenbauer polynomials are orthogonal on [-1,1] with respect to weight . We especially need in the Gegenbauer polynomials with v . They are orthogonal with respect to the weight

$$\omega_q(u) = \omega_{\frac{q-3}{2},\frac{q-3}{2}}(u) = (1-u^2)^{\frac{q-3}{2}}$$

Or, equivalent, to

$$\Omega_q(u) = \frac{\omega_q(u)}{\tau_q}$$

Where

$$\tau_q = \int_{-1}^{1}\omega_q(u)\,du = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{m-1}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$$

The Cristoffel-Darboux kernel which relates to the Jacobi polynomials is

$$K_t^{(\alpha,\beta)}(u,v) = \sum_{k=0}^{t}\frac{P_k^{(\alpha,\beta)}(u)P_k^{(\alpha,\beta)}(v)}{\|P_k^{(\alpha,\beta)}\|^2_{\omega_{\alpha,\beta}}}.$$

According to the Cristoffel – Darboux Formula

$$K_t^{(\alpha,\beta)}(u,v) = \frac{1}{2^{\alpha+\beta}(2t+\alpha+\beta+2)}\cdot\frac{\Gamma(t+2)\Gamma(t+\alpha}{\Gamma(t+\alpha+1)\Gamma(} \times \frac{P_{t+1}^{(\alpha,\beta)}(u)P_t^{(\alpha,\beta)}(v) - P_t^{(\alpha,\beta)}(u)P_{t+1}^{(\alpha,\beta)}(v)}{u-v}.$$

An important particular case is

$$K_t^{(\alpha,\beta)}(u) \equiv K_t^{(\alpha,\beta)}(u,1) = \frac{1}{2^{\alpha+\beta+1}}\cdot\frac{\Gamma(t+\alpha+\beta+2}{\Gamma(\alpha+1)\Gamma(t+\beta+}$$

Whence

$$K_t^{(\alpha,\beta)}(1) = \frac{1}{2^{\alpha+\beta+1}}\cdot\frac{\Gamma(t+\alpha+\beta+2)\Gamma(t+\alpha+2)}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(t+1)\Gamma(t+\beta+1)}.$$

In fact, we need to calculate the quantity

$$\Lambda_t^{(\alpha,\beta)} = 2^{\varepsilon}\tau_{\alpha,\beta}K_{[\frac{t}{2}]}^{(\alpha,\beta+\varepsilon)}(1)$$

RESEARCH METHODOLOGY

Polynomial Functions

E is supposed to be a m-dimensional right linear Euclidean space over the field K = R or COMPUTER, or H. We will deal with complex-valued polynomial functions on the real unit sphere S(E) and the projective space P(E).

Polynomial Functions on the real unit spheres

The unit sphere S(E) is a real algebraic manifold.

In spirit of Algebric Geometry, we define a polynomial function as a restriction to S(E) of a polynomial

Given a basis in E, the general form of polynomials on E is

$$\psi(x) = \sum_{I} \alpha_I [\xi]^I$$

Where I runs over a finite set of multiindices are the corresponding monomes with respect to the coordinates.

$$|I| = \sum_{k=1}^{m} i_k$$

Deg is the maximal value of the |I|. This number is independent of the choice of basis since for given coefficients are uniquely determined. At least one of them is different from zero. The set P(E) of all polynomials on E is a linear space with respect to the standard linear operations in functional spaces. P(E) is a ring so that P(E) is an algebra over C. Hence, the set Pol (E) of all polynomial functions on S(E) is also a linear space.

$$r\psi = \psi|\mathbf{S}(E)$$

There is a lot of polynomials which generate the same polynomial functions. The point is that the kernel of the homomorphism r is the subspace.

$$\mathcal{P}^0(E) = \{\psi \in \mathcal{P}(E) : \psi(x) = (1 - \|x\|^2)\omega(x),\ \omega \in \mathcal{P}(E)$$

Note that a definite lifting does exist if the polynomial is homogeneous,

$$\psi(x\gamma) = \gamma^d \psi(x), \quad \gamma \in \mathbf{R},$$

Where d= deg implies,

$$\psi(x) = \|x\|^d \phi(\widehat{x}), \quad x \in E.$$

This formula is known as Homogeneous Lifting. In the non – homogeneous case,

$$\psi(x) = \sum_{k=0}^{d} \psi_k(x), \quad x \in E,$$

By restriction, we get

$$\phi(x) = \sum_{k=0}^{d} \phi_k(x), \quad x \in \mathbf{S}(E),$$

Then

$$\psi(x) = \sum_{k=0}^{d} \|x\|^k \phi_k(x), \quad x \in E.$$

In this way, we recover the polynomial as soon as all homogeneous components are given. In order to overcome this difficulty, we have to restrict the space P(E) to its subset H(E),

$$\mathcal{H}(E) = \{\psi \in \mathcal{P}(E) :\ \Delta\psi = 0\}.$$

NEED OF STUDY

Projective Codes, Cubature Formulas and Designs

A Projective Code X is a spherical code such that the Projectivization is objective or the points from X are protectively distinct. It is convenient for any spherical code Y to treat its Projectivization. For a Projective Code X its angle set is defined as

$$a(X) = \{2|\langle x,y\rangle|^2 - 1 :\ x,y \in X,\ \ x \neq y\} \subset [-1,1)$$

THEOREM : Let X be a projective code, | X | = n, | a(X) | = s. Then

$$n \leq \begin{cases} \binom{m+2s-1}{m-1} & (\mathbf{K}=\mathbf{R}) \\ \binom{m+s-1}{m-1}^2 & (\mathbf{K}=\mathbf{C}) \\ \frac{1}{2m-1}\binom{2m+s-2}{2m-2}\cdot\binom{2m+s-1}{2m-2} & (\mathbf{K}=\mathbf{H}) \end{cases}$$

Proof : Consider the polynomial f, deg f =s, such that f|a(X) = 0. Then f(1) != 0. We get

$$n \leq \sum_{k=0}^{s} h_{m,2k}$$

$$n \leq \dim \mathrm{Pol}_{\mathbf{K};2s}(E)$$

DEFINITION : A Projective Cubature Formula of index 2t is an identity

$$\int \phi \, d\varrho = \int \phi \, d\sigma, \quad \phi \in \mathrm{Pol}_{\mathbf{K}}(E; 2t)$$

And

$$\int \phi \, d\varrho \equiv \int_{\mathrm{supp}\varrho} \phi \, d\varrho = \sum_{x \in \mathrm{supp}\varrho} \phi(x)\varrho(x) \equiv \sum_{x} \varphi(x)\varrho(x).$$

The set suppq is called the support of the projective cubature formula. Note that suppq is podal in the real case. Actually, a real projective cubature formula of index 2t is the same as a podal spherical cubature formula of index 2t.

$$\mathrm{Pol}_{\mathbf{K}}(E; 2t) \subset \mathrm{Pol}(E; 2t) \equiv \mathrm{Pol}_{\mathbf{R}}(E_{\mathbf{R}}; 2t).$$

The identity can be rewritten as:

$$\sum_{k=1}^{n} \phi(x_k)\varrho_k = \int \phi \, d\sigma, \quad \phi \in \mathrm{Pol}_{\mathbf{K}}(E; 2t),$$

SCOPE OF RESEARCH

Isometric Embeddings

The number p must be even integer otherwise such an embedding could not exist according to Theorem. We start with decomposition. The isometry property in the coordinate form is equivalent to

$$\sum_{j=1}^{n} |\langle u_j, x\rangle|^p = \langle x, x\rangle^{\frac{p}{2}}, \quad x \in \mathbf{K}^m$$

This basis identity can be rewritten as

$$\sum_{j=1}^{n} |\langle u_j, x\rangle|^p = \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right) \cdot \int |\langle y, x\rangle|^p \, d\sigma(y), \quad x \in \mathbf{K}^m,$$

However, the vectors cannot be normalized. It is possible, that there is a pair of proportional non zero vectors. Then,

$$|\langle u_1, x\rangle|^p + |\langle u_2, x|^p = |\langle \tilde{u}_1, x|^p$$

Where

$$\tilde{u}_1 = u_1(1 + |\gamma|^p)^{\frac{1}{p}}$$

There exists a close relation between isometric embeddings and projective cubature formula of index p.

THEOREM : An isometric embedding exists if and only if there exists a projective cubature formula of index p.

Proof : Suppose that an isometric embedding exists. We obtain the projective cubature formula of index p with the nodes and the weights

$$x_j = \frac{\tilde{u}_j}{\|\tilde{u}_j\|} \in S(\mathbf{K}^m), \quad \varrho_j = \frac{\|\tilde{u}_j\|^p}{\Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)}, \quad 1 \le j \le \nu,$$

Then, we get

$$u_j = \left(\varrho_j \Upsilon_{\mathbf{K}}\left(m, \frac{p}{2}\right)\right)^{\frac{1}{p}} x_j, \quad 1 \le j \le \nu; \quad u_j = 0, \quad \nu < j \le n.$$

THEOREM: There exists a quaternionic projective formula of index 10 with 6486480 nodes on S(H).

COROLLARY : The inequality

NH(7 , 10) < = 6486480 holds.

THEOREM : There exist the quaternionic projective cubature formulas of index 4 with

$m = 2^{2k-2} + q + 1$, $n = 2^{2k+2} . 3^{4q+1} . (2^{2k-1})$, k>=1, q>=0

or with

$m = 2k+q+2$, $n=3^{q+1} ((k+1)^2 + 1)$

where

k is prime power , q>=0 , 2k + q + 2 == 0 (mod 4)

Invariant Cubature Formulas

Here, we consider the cubature formulas which are invariant with respect to a group action. Let G be a finite subgroup of the unitary group U (E). There is a natural action of G on S(E),

X → gx,

A spherical code is called G – invariant if GX = X. For every point, its orbit Gx is the minimal G- invariant spherical code containing x. A G-invariant spherical code X is called G-homogeneous if it is an orbit , i.e. the action is transitive.

For any spherical code V, the Orbit GV is the minimal G-invariant spherical code containing V.

A measure q is called G- invariant if gq = q for all g, or, in other words , the set supp q is G – invariant .

DEFINITION: A spherical cubature formula is called G – invariant if the measure q is G – invariant.

Obviously, if a spherical cubature formula is G – invariant and its support is G- homogeneous then the support is a spherical design.

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