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An Investigation about Classical and Modern Game Theory: A Case Study of Evolutionary and New Game Theory

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Abstract – This study begins with some basic terminology, introducing elementary game theoretic notions such as strategy, best reply, Nash equilibrium pairs etc. Players who use strategies which are in Nash equilibrium have no incentive to deviate unilaterally. Next, a population viewpoint is introduced. The simplest description of such an evolution is based on the replicator equation. The relation between Nash equilibria and rest points of the replicator equation are investigated, which leads to a short proof of the existence of Nash equilibria. We then study mixed strategies and evolutionarily stable strategies. This introductory section continues with a brief discussion of other game dynamics, such as the best reply dynamics, and ends with the simplest extension of replicator dynamics to asymmetric games.

Evolutionary game theory has grown into an active area of research that bridges concepts from biology, evolution, non-linear dynamics, and game theory. The mechanisms necessary to conduct an evolutionary analysis of games are presented. Relations between evolutionary stable strategies and Nash equilibria are considered. Replicator dynamics are developed and applied to three relevant games. The analysis of example games is used to illustrate the weaknesses and strengths of the theory.

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INTRODUCTION

Is there progress in game theory? Do we know more today in this field than the scholars in the decade after John von Neumann and Oskar Morgenstern published their pioneering *Game Theory and Economic Behavior* in 1944. Or, did we only experience a change in style and language over the last fifty years. The hypothesis of the following brief history of game theory is that the various stages of development are the result of different assumptions about the nature of the decision makers underlying the alternative game theoretical approaches. The following text will not give a historical overview which aims for completeness.

Rather, it will trace the changes in the "image of man" implicit in the development of game theory and demonstrate some of consequences that follow.

We will distinguish three major stages in the development of game theory. The first one, *classical game theory*, is defined by John von Neumann's and Oskar Morgenstern's book. It introduced axioms for the concept of the individual rational player. Such a player makes consistent decisions in the face of certain and uncertain alternatives. But, such a player does not necessarily assume that other players also act rationally. In contrast, *modern game theory* is defined by the *Nash player* who is not only rational but assumes that all players are rational

to such a degree that they can coordinate their strategies so that a Nash equilibrium prevails. The more recent, third stage in the development of game theory, *new game theory*, is defined by the *Harsanyi player*. This player is rational but knows very little about the other players, e.g., their payoff functions or the way they form beliefs about other players' payoff functions or beliefs. This limitation initiated two complementary strings of research: the more traditional one, based on a rational choice model, is characterized by the analysis of interactive *gedankenexperiments* about forming beliefs (i.e., epistemic games), while the second string follows an evolutionary approach where the agents rest content with themselves by imitating the observed successful behavior of other agents. The latter can be interpreted as the "rational conclusion" of the constrained cognitive capacity of the decision maker, on the one hand, and the complexity of the decision situation, on the other, or seen as the consequence suggested by the results of empirical research which challenge the rational choice model and its teleological background.

Evolutionary game theory studies the behavior of large populations of agents who repeatedly engage in strategic interactions. Changes in behavior in these populations are driven either by natural selection via differences in birth and death rates, or by the application of myopic decision rules by individual agents.

The birth of evolutionary game theory is marked by the publication of a series of papers by mathematical biologist John Maynard Smith. Maynard Smith adapted the methods of traditional game theory, which were created to model the behavior of rational economic agents, to the context of biological natural selection.

Towards the end of this period, economists realized the value of the evolutionary approach to game theory in social science contexts, both as a method of providing foundations for the equilibrium concepts of traditional game theory, and as a tool for selecting among equilibria in games that admit more than one. Especially in its early stages, work by economists in evolutionary game theory hewed closely to the interpretation set out by biologists, with the notion of ESS and the replicator dynamic understood as modeling natural selection in populations of agents genetically programmed to behave in specific ways. But it soon became clear that models of essentially the same form could be used to study the behavior of populations of active decision makers.

While the majority of work in evolutionary game theory has been undertaken by biologists and economists, closely related models have been applied to questions in a variety of fields, including transportation science, computer science, and sociology. Some paradigms from evolutionary game theory are close relatives of certain models from physics, and so have attracted the attention of workers in this field. All told, evolutionary game theory provides a common ground for workers from a wide range of disciplines.

CLASSICAL GAME THEORY AND THE AUTONOMOUSLY RATIONAL PLAYER

Game theorists consider the axiomatization of the utility function in the case of uncertainty a major contribution in von Neumann and Morgenstem (1944). It paved the ground for the modeling of rational decision-making when a decision maker is faced by lotteries. Thereafter a utility function. $u_i(.)$. which satisfies the expected utility hypothesis, i.e.

$$u_i([A,p;B,1\text{-}p]) = pu_i(A) + (1\text{-}p)u_i(B) \qquad (1)$$

is called a von Neiunann-Morgenstem utility function. In (1). A and B are events (or alternatives), p is the probability' that event A occurs while 1-p is the probability of B occuring. Thus $[A,p;B,1\text{-}p]$ is a lottery (or prospect). It is a notational convention to write $[A,p;B,1\text{-}p] = A$ if p = 1 and $[A,p;B,1\text{-}p] = B$ if $p = 0$. Of course, $[A,p;X,1\text{-}p] = A$ for every alternative X if p = 1.

The probabilities p can be related to a model of relative frequencies and are. in this sense, objective and thus represent risk: or they can be subjective (i.e. expectations or beliefs) and thus represent uncertainty. The classical distinction between risk and uncertainty going back to Frank Knight (1921) appears, however, to be somewhat outdated today. For it does not seem to really matter in the end whether we believe in the objectivity of relative frequencies as an outcome of a random mechanism, or whether we derive our expectations from introspection and *gedankenexperiments.* One way or the other, they are all based on beliefs which reflect uncertainty and thus are subjective. If we follow this view and define rational behavior under uncertainty as maximizing expected utility in terms with (1), then our approach is *Bayesian.*

The utility values which the function $u_i(.)$ assigns to events (such as money, cars, or strawberries) are called payoffs. Because of (1) we do not have to distinguish between payoff and expected payoffs: if player i is indifferent between the lottery $[A,p;B,1\text{-}p]$ and the sure event C then $u_i([A,p;B,1\text{-}p]) = u_i(C)$, i.e.. the payoffs are identical. If $u_i(.)$ satisfies (1) then it is well defined as utility function of individual i up to a linear order-preserving transformation. That is. if $v_i(.) = a_iu_i(.) + b_i$ and $a_i > 0$ then $u_i(.)$ and $v_i(.)$ represent identical utility functions: thus $u_i(.)$ defines not a function but a family of functions and interpersonal comparison of utility is excluded because a_i and b_i are not determined.

The utility function of individual i can be linear, concave or convex in money - which coincides with risk neutrality, risk aversion, and risk affinity in so far as money defines the events of a lottery - or $u_i(.)$ can be related with money in a less rigid way without violating (1). There is. however, ample empirical evidence that individual behavior does normally not follow a pattern which is consistent with (1). There are also strong intuitive arguments which challenge the adequacy of individual axioms which underlie the theory expressed in (1) such as the so-called *Allais paradox.* Later Nobel Laureate Maurice Allais (1953) demonstrated the proposed inconsistency of the axioms of the von Neumann Morgenstem utility' theory by means of the following example:

(1) People are asked whether they prefer alternative A or alternative B

where

Alternative A: 100 million for sure

a chance of 0.1 to win 500 millions

Alternative B: a chance of 0.89 to win 100 millions

a chance of 0.01 to win nothing

(2) People are asked whether they prefer alternative C or alternative D

where

Alternative C: a chance of 0.11 to win 100 million *a* chance of 0.89 to win nothing

Alternative D: a chance of 0.1 to win 500 million a chance of 0.9 to win nothing

The money values are probably in "old" French francs. The expected values of A. B. C. and D are (measured in millions) 100.139.11 and 50, respectively.

Allais argues that for a large number of people, especially for those who are averse against taking risk, one observes that they prefer A to B and D to C. However, von Neumann Morgenstem utility theory suggests that if A is preferred B then C is preferred to D. In order to see this, we write these preference relations in terms of the von Neumann Morgenstem utility function of an agent i:

"A preferred to B" implies: $u_i(100) > 0.1u_i(500) + 0.89u_i(100) + 0.01u_i(0)$

"C preferred to D" implies: $0.11u_i(100) + 0.89\ u_i(0) > 0.1u_i(500) + 0.9u_i(0)$

Both inequalities can be reduced to $0.11u_i(100) > 0.1u_i(500) + 0.01u_i(0)$. Tints

"A preferred to B" implies "C preferred to D". Consequently. "D preferred to C" is inconsistent with "A preferred to B" and corresponding behavior violates the expected utility hypothesis (1).

There are. however, also strong arguments in favor of (1) and the underlying axioms formalized in von Neumann and Morgenstem (1944). Firstly.

there is empirical evidence that people tend to correct their behavior if they are aware that it deviates from (1) or one of its implications. Secondly, the generalization of alternative approaches to decision-making under uncertainty (such as the *prospect theory* of Kahneman and Tversky (1979) and the *similarity approach* of Rubinstein (1998)) are also criticized on the basis of contradicting empirical results and implausibility of underlying assumptions. Moreover, the alternative approaches tend to be more complicated than the theory behind (1) and therefore more difficult to apply to real life decision-making and textbook analysis. This is perhaps the main reason why game theorists stick to the von Neiunann-Morgenstem utility function when it comes to decision-making under uncertainty. The maximization of such a utility function defines the *rational player* in game situations, i.e. if the outcome of a choice depends on the action of at least two agents and the agents, in principle, put themselves into the shoes of the other agents when they make their decisions because they know of the interdependence of decisions.

There are however many ways to specify this knowledge and thus the image which a player has of the other player(s). Von Neumann and Morgenstem (1944) assumed that a player i does not expect that player j is necessarily rational: j's behavior may violate the theory embedded in (1) and its implications. In their theory of games, they propose that players should act rational even under the assumption that other players are irrational, i.e. inconsistent with (1):"... the rules of rational behavior must provide definitely for the possibility of irrational conduct on the part of others.... In whatever way we formulate the guiding principles and the objective justification of' rational behavior.' provision will have to be made for every possible conduct of' the others'". To characterize this proposition we will speak of *autonomously rational players* in the theory of von Neumann and Morgenstem.

The Minimax Theorem -

It may come somewhat of a surprise, but von Neumann and Morgenstem's theory provides convincing results only if we have a situation in which there is *pure conflict of interest* between two players and the decision situation can be modeledas a zero-sum game. For example, if we assume that the payoff (bi-)matrix in Figure 1 is specified by the payoff values a = $-\alpha$. b =$-\beta$, c =$-\gamma$, and d = $-\delta$. then it describes a zero-sum (two -by-two) game where player 1 has the pure strategies s_{11} and s_{12} and player 2 has the pure strategies s_{21} and s_{22}

	s_{21}	s_{22}
s_{11}	(a,α)	(b,β)
s_{12}	(c,γ)	(d,δ)

Figure 1: Generalized two -by-two game

In principle, the definition of utility functions given does not allow for interpersonal comparison of utility as implied by the zero-sum property. However, if there is pure conflict of interest between two players then the assumption that a utility gain to player 1 is a utility loss to player 2. and vice versa, seems appropriate. Note that, if the payoff values of the two players in each cell add to the same constant value, then the game is equivalent to a zero-sum same and can. without loss of information, be transformed into such a same.

Given a zero-sum game, von Neumann and Morgenstem (1944) suggest that each player will choose his maximin strategy. Thus player 1 looks for the minimum payoff in each line and then earmarks the strategy which is related to the highest payoff of these (two) minima while player 2 does likewise for his payoffs in each column. If the earmarked value of player 1 and the earmarked value of player 2 add up to zero, then the corresponding strategy pair characterizes the solution and the related payoff pair describes the outcome.

If the earmarked values do not add up to zero, then player i (i = 1. 2) will randomize on his strategies such that the expected value is independent of whether the other player chooses his first or second strategy or any mixture of these strategies. For instance, if player 1 chooses his first strategy with probability p and player 2 chooses his first strategy with probability q. then p and q are determined by the two equalities:

$$pa + (1-p)c = pb + (1-p)d \text{ and } q\alpha + (1-q)\beta = q\gamma + (1-q)\delta$$

Solving these equalities, we get

$$p^{\circ} = \frac{d-c}{a-b-c+d} \text{ and } q^{\circ} = \frac{\delta-\beta}{\alpha-\beta-\gamma+\delta} \qquad (2)$$

It is easy to show that the (expected) payoff player 1 is equal to the negative of the payoff of player 2 if they choose their corresponding first strategies with probabilities p° and q°. This is the essence of the so-called *minimax theorem* of von Neumann and Morgenstem which says that, given a two-person zero-sum game, there is *always* a pair of strategies, either in pure or mixed strategies, such that the maximin payoff equals the minimax payoff of player 1. Note that in two- person zero-sum games the maximin payoff of player 2 with respect to his own payoff values is identical to the minimax value with respect to the payoffs of player 1. (Because the payoffs of player 2 are the negative values of the payoffs of player 1. it is sufficient to specify the payoffs of player 1 only.)

LIMITATIONS OF CLASSICAL GAME THEORY-

Baumol (1972. p. 575) summarizes the classical view on game theory which derives from the minimax theorem: "In game theory, at least in the zero-sum. two-person case, there is a major element of predictability in the behavior of the second player. He is out to do everything he can to oppose the first player. If he knows any way to reduce the first player's payoff, he can be counted upon to employ it." However, the *minimax theorem* loses its power if the players' interests do not contain pure conflict and the zero-sum modeling becomes inappropriate. This is particularly the case if strategic coordination problems become eminent. For instance, let's assume that a > 0. α>0. d > 0. δ>0. and all other payoffs in Figure 1 are zero. Then the matrix in Figure 1 represents a *variable-sum game* and the minimax theorem does not. in general, apply anymore. Assume further, player 1 and 2 have to choose their strategies simultaneously - or in such a way such that they cannot see what the other player has chosen. Then a player has to solve the problem of how to coordinate his strategy with the strategy of the other player hi order to gain positive payoffs. The fact that the theory of von Neumann and Morgenstem says little about coordination problems, in particular, and variable-sum games, in general, concurs with the problem that the guiding hand of self-interest becomes weak in strategic situations if there is no pure conflict of interest and players have difficulties to form expectations about the behavior of their fellow players.

It is not surprising that the textbook representation of game theory of the 1950s and still in the early 1960s focused on the two-person zero-stun game and problems of how to calculate the maximin solution if players have more than two pure strategies. .An exception is the ingenious book by Luce and Raiffa (1957) which is still a great source of inspiration for game theorists.

The assumption of a pure conflict of interest seems also questionable if there are more than two players. If we try to formulate a zero-sum game for three players then the problem becomes rather obvious. Moreover. 111 the case of more than two players there is a potential for coalitions. Von Neumann and Morgenstem (1944) developed the concept of the characteristic function in order to express the value of a coalition. They also suggested a solution concept for the case of more than two players which may take care of coalition formation: they simply called this concept *solution*, However, neither does its application give an answer which coalition will form nor does it determine the payoffs which the individual players get in the course of the game.

It is fair to mention that even more than fifty years later the existing theories of coalition formation provide answers to these two problems only if the coalition games are rather specific and the competing theories generally provide divergent answers. However, for the case of variable-sum games and games with more than two players (if they do not form coalitions) a very promising solution concept has been suggested: the Nash equilibrium and its refinements.

NASH EQUILIBRIUM

Pure-Strategy Nash Equilibrium -

Rational players think about actions that the other players might take. In other words, players form beliefs about one another's behavior. For example, in the BoS game, if the man believed the woman would go to the ballet, it would be prudent for him to go to the ballet as well. Conversely, if he believed that the woman would go to the fight, it is probably best if he went to the fight as well. So, to maximize his payoff, he would select the strategy- that yields the greatest expected payoff given his belief. Such a strategy- is called a best response (or *best reply).*

Definition 1. Suppose player *i* has some belief $s_{-i} \in S_{-i}$ about the strategies played by the other players. Player *Vs* strategy $s_i \in S_i$ is a best response if

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \text{ for every } s_i' \in S_i.$$

We now define the best response correspondence), $BR_i(s_{-i})$, as the set of best responses player i has to s_{-i}. It is important to note that the best response correspondence is setvalued. That is, there may be more than one best response for any given belief of player *i.* If the other players stick to s_{-i}, then player *i* can do no better than using any of the strategies in the set $BR_i(s_{-i})$. In the BoS game, the set consists of a single member: $BR_m(F) = \{F\}$ and $BR_m(B) = \{B\}$. Thus, here the players have a single optimal strategy for every belief. In other games, like the one in Fig. 2, $BR_i(s_{-i})$ can contain more than one strategy-.

In this game, $BR_1(L) = \{M\}$, $BR_1(C) = \{U, M\}$, and $BR_1(R) = \{U\}$. Also, $BR_2(U) = \{C, R\}$ $BR_2(M) = \{R\}$, and $BR_2(D) = \{C\}$. You should get used to thinking of the best response correspondence as a set of strategies, one for each combination of the other players' strategies. (This is why we enclose the values of the correspondence in braces even when there is only- one element.)

		Player 2 L	Player 2 C	Player 2 R
Player 1	U	2,2	1,4	4,4
Player 1	M	3,3	1,0	1,5
Player 1	D	1,1	0,5	2,3

Figure 2: The Best Response Game.

We can now use the concept of best responses to define Nash equilibrium: a Nash equilibrium is a strategy profile such that each player's strategy is a best response to the other players' strategies:

Definition 2 (Nash Equilibrium). The strategy profile $(s_i^*, s_{-i}^*) \in S$ is a pure-strategy Nash equilibrium if, and only if, $s_i^* \in BR_i(s_{-i}^*)$ for each player $i \in I$.

An equivalent useful way of defining Nash equilibrium is in terms of the payoffs players receive from various strategy profiles.

Definition 3. The strategy profile (s_i^*, s_{-i}^*) is a pure-strategy Nash equilibrium if, and only if, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for each player $i \in I$ and each $s_i \in S_i$.

That is, for every player *i* and every strategy s_i of that player, (s_i^*, s_{-i}^*) is at least as good as the profile (s_i, s_{-i}^*) in which player *i* chooses s_t and every other player chooses s_{-i}^*. In a Nash equilibrium, no player *i* has an incentive to choose a different strategy when everyone else plays the strategies prescribed by the equilibrium. It is quite important to understand that a strategy profile is a Nash equilibrium if no player has incentive to deviate from his strategy given that the other players do not deviate. When examining a strategy- for a candidate to be part of a Nash equilibrium (strategy- profile), we always hold the strategies of all other players constant.

To understand the definition of Nash equilibrium a little better, suppose there is some player *i,* for whom 5, is not a best response to s_{-i}. Then, there exists some s_i' such that $u_i(s_i', s_{-i}) > u_i(s_i, s_{-i})$. Then this (at least one) player has an incentive to deviate from the theory's prediction and these strategies are not Nash equilibrium.

Another important thing to keep in mind: Nash equilibrium is a strategy profile. Finding a solution to a game involves finding strategy- profiles that meet certain rationality requirements. In strict dominance we required that none of the players' equilibrium strategy is strictly dominated. In Nash equilibrium, we require that each player's strategy is a best response to the strategies of the other players.

The Prisoner's Dilemma. By examining all four possible strategy profiles, we see that *(D, D)* is the unique Nash equilibrium (NE). It is NE because (a) given that player 2 chooses *D,* then player 1 can do no better than chose *D* himself (1 >0); and (b) given that player 1 chooses *D,* player 2 can do no better than choose *D* himself. No other strategy profile is NE:

- (C, C) is not NE because if player 2 chooses C, then player 1 can profitably deviate by-choosing *D* (3 > 2). Although this is enough to establish the claim, also note that the profile is not NE for another sufficient reason: if player 1 chooses C, then player 2 can profitably deviate by playing *D* instead. (Note that it is enough to show that one player can deviate profitably for a profile to be eliminated.)

- (*C,D*) is not NE because if player 2 chooses *D,* then player 1 can get a better payoff by choosing *D* as well.

- (D, C) is not NE because if player 1 chooses *D,* then player 2 can get a better payoff by choosing *D* as well.

Since this exhausts all possible strategy profiles, *(D,D)* is the unique Nash equilibrium of the game. It is no coincidence that the Nash equilibrium is the same as the strict dominance equilibrium we found before. In fact, as you will have to prove in your homework, a player will never use a strictly dominated strategy in a Nash equilibrium. Further, if a game is dominance solvable, then its solution is the unique Nash equilibrium.

How do we use best responses to find Nash equilibria? We proceed in two steps: First, we determine the best responses of each player, and second, we find the strategy profiles wiiere strategies are best responses to each other.

For example, consider again the game in Fig. 2. We have already determined the best responses for both players, so we only need to find the profiles where each is best response to the other. An easy way to do this in the bi-matrix is by going through the list of best responses and marking the payoffs with a for the relevant player where a profile involves a best response. Thus, we mark player l's payoffs in (*U,C*), (*U,R*), (M,L), and (M,C). We also mark player 2's payoffs in (*U,C*), (*U,R*), (*M,R*), and (D,C). This yields the matrix in Fig. 3.

		Player 2		
		L	*C*	*R*
Player 1	*U*	2,2	1*,4*	4*,4*
	M	3*,3	1*,0	1,5*
	D	1,1	0,5*	2,3

Figure 3: The Best Response Game Marked.

There are two profiles with stars for both players, (*U,C)* and (*U,R*), which means these profiles meet the requirements for NE. Thus, we conclude this game has two pure-strategy Nash equilibria.

Strict Nash Equilibrium-

Consider the game in Fig. 5. Its story goes like this. The setting is the South Pacific in 1943. Admiral Kimura has to transport Japanese troops across the Bismarck Sea to New Guinea, and Admiral Kenney wants to bomb the transports. Kimura must choose between a shorter Northern route or a longer Southern route, and Kenney must decide where to send his planes to look for the transports. If Kenney sends the plans to the wrong route, he can recall them, but the number of days of bombing is reduced.)

This game has a unique Nash equilibrium, in which both choose the northern route, *(N, N).* Note, however, that if Kenney plays N, then Kimura is indifferent between *N* and *S* (because the advantage of the shorter route is offset by the disadvantage of longer bombing raids). Still, the strategy profile (*N,N)* meets the requirements of NE. This equilibrium is not strict.

More generally, an equilibrium is strict if, and only if, each player has a unique best response to the other players' strategies:

Definition 3.4. A strategy profile (s_i^*, s_{-i}^*) is a strict Nash equilibrium if for every player i, $u_i(s_i^*, s_{-i}^*) > u_i(s_i, s_{-i}^*)$ for every strategy $s_i \neq s_i^*$ The difference from the original definition of NE is only in the strict inequality sign.

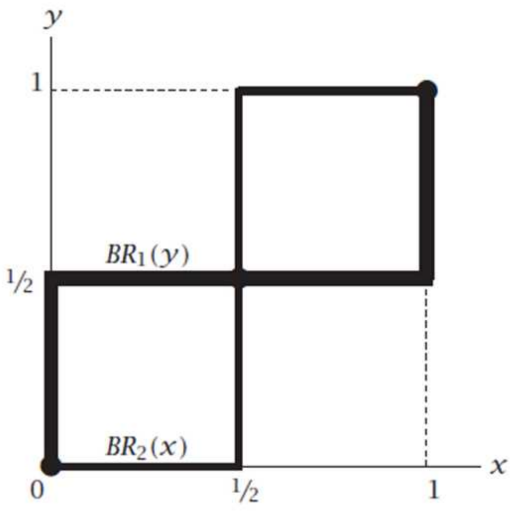

Figure 4: Best Responses in the Modified Partnership Game.

		Kimura: N	Kimura: S
Kenney	N	2, −2	2, −2
	S	1, −1	3, −3

Figure 5: The Battle of Bismarck Sea.

Mixed Strategy Nash Equilibrium-

The most common example of a game with no Nash equilibrium in pure strategies is Matching Pennies, which is given in Fig. 6.

		Player 2: H	Player 2: T
Player 1	H	1, −1	−1, 1
	T	−1, 1	1, −1

Figure 6: Matching Pennies.

This is a strictly competitive (zero-sum) situation, in which the gain for one player is the loss of the other. This game has no Nash equilibrium in pure strategies. Let's consider mixed strategies.

We first extend the idea of best responses to mixed strategies: Let $BR_i(\sigma_{-i})$ denote player *i's* best response correspondence when the others play σ_{-i}. The definition of Nash equilibrium is analogous to the pure-strategy- case:

Definition 5. A mixed strategy profile σ^* is a mixed-strategy Nash equilibrium if, and only if, $\sigma_i^* \in BR_i(\sigma_{-i}^*)$.

As before, a strategy profile is a Nash equilibrium whenever all players' strategies are best responses to each other. For a mixed strategy- to be a best response, it must put positive probabilities only on pure strategies that are best responses. Mixed strategy- equilibria, like pure strategy equilibria, never use dominated strategies.

Turning now to Matching Pennies, let $\sigma_1 = (p, 1-p)$ denote a mixed strategy for player 1 where he chooses *H* with probability p, and *T* with probability-$1-p$. Similarly, let $\sigma_2 = (q, 1-q)$ denote a mixed strategy- for player 2 where she chooses *H* with probability q, and *T* with probability-$1-q$. We now derive the best response correspondence for player 1 as a function of player 2's mixed strategy-.

Player I's expected payoffs from his pure strategies given player 2's mixed strategy are:

$$U_1(H, \sigma_2) = (1)q + (-1)(1-q) = 2q - 1$$
$$U_1(T, \sigma_2) = (-1)q + (1)(1-q) = 1 - 2q.$$

Playing *H* is a best response if, and only if:

$$U_1(H, \sigma_2) \geq U_1(T, \sigma_2)$$
$$2q - 1 \geq 1 - 2q$$
$$q \geq 1/2.$$

Analogously, *T* is a best response if, and only if, $q \leq 1/2$. Thus, player 1 should choose *p* = 1 if $q \geq 1/2$ and $p = 0 \text{ if } q \leq 1/2$. Note now- that whenever $q = 1/2$, player 1 is indifferent between his two pure strategies: choosing either one yields the same expected payoff of 0. Thus, both strategies are best responses, which implies that any mixed strategy that includes both of them in its support is a best response as well. Again, the reason is that if the player is getting the same expected payoff from his two pure strategies, he will get the same expected payoff from any mixed strategy- whose support they are.

Analogous calculations yield the best response correspondence for player 2 as a function of σ_1. Putting these together yields:

$$BR_1(q) = \begin{cases} 0 & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q > 1/2 \end{cases} \qquad BR_2(p) = \begin{cases} 0 & \text{if } p > 1/2 \\ [0,1] & \text{if } p = 1/2 \\ 1 & \text{if } p < 1/2 \end{cases}$$

The graphical representation of the best response correspondences is in Fig. 7. The only place where the randomizing strategies are best responses to each other is at the intersection point, where each player randomizes between the two strategies with probability $1/2$. Thus, the Matching Pennies game has a unique Nash equilibrium in mixed strategies $\langle\sigma_1^*, \sigma_2^*\rangle$, where $\sigma_1^* = (1/2, 1/2)$, and $\sigma_2^* = (1/2, 1/2)$. That is, where $p = q = 1/2$.

As before, the alternative definition of Nash equilibrium is in terms of the payoff functions. We require that no player can do better by using any other strategy- than the one he uses in the equilibrium mixed strategy profile given that all other players stick to their mixed strategies. In other words, the player's expected payoff of the MSNE profile is at least as good as the expected payoff of using any other strategy.

Definition 6. A mixed strategy- profile σ^* is a mixed-strategy Nash equilibrium if, for all players i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(s_i, \sigma_{-i}^*) \text{ for all } s_i \in S_i.$$

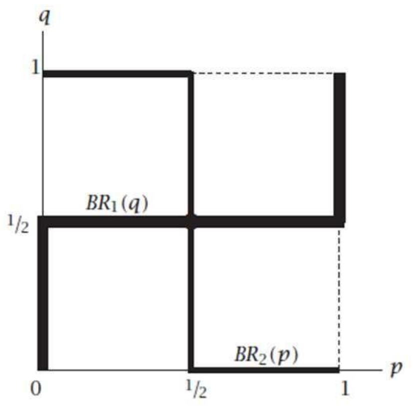

Figure 7: Best Responses in Matching Pennies.

Since expected utilities are linear in the probabilities, if a player uses a non-degenerate mixed strategy in a Nash equilibrium, then he must be indifferent between all pure strategies to which he assigns positive probability. This is why we only need to check for a profitable pure strategy deviation.

EVOLUTIONARY GAME THEORY

Evolutionary game theory (EGT) has grown into a field that combines the principles of game theory, evolution, and dynamical systems to interpret the interactions of biological agents. Practitioners in the field have used the theory to explain biological phenomena successfully, but EGT can also be used to interpret classical games from a different perspective. This document introduces evolutionary game theory and presents an evolutionary approach to the analysis of games.

There are several basic components in the EGT analysis of games. Game agents and their strategies must be simulated with populations of players, the fitness of different strategies relative to the population must be computed, and a process to govern the evolution of the population must be defined.

These simple components can be combined to yield highly complex solutions. Ideally, under the dynamical process the strategies of the populations of players will converge to some stable value. Evolutionary game theorists often claim the evolutionary solution of the game as the true definition of rational play.

The concept of simulating populations of players to determine rational play is not new. A similar idea was apparently suggested by Nash in his doctoral thesis. The real birth of EGT, though, is likely due to Maynard Smith . Work by Cressman focusses heavily on the stability analysis of games. In Weibul a broader treatment of continuous and discrete replicator dynamics is given, but with few applications. Recent work by Hofbauer and Sigmund provide an excellent mathematical treatment of the topic with many examples .

Evolutionary game theory is a different approach to the classic analysis of games. Instead of directly calculating properties of a game, populations of players using different strategies are simulated and a process similar to natural selection is used to determine how the population evolves. Varying degrees of complexity are required to represent populations in multi-agent games with differing strategy spaces.

To be exact, consider a ?r-plaver game where the i^{th} player has strategy space denoted by S_i. An EGT approach would be to model each agent by a population of players. The population for the i^{th} agent would then be partitioned into groups $E_{i1}, E_{i2}, \ldots, E_{ik}$ ***(k*** might be different for each population). Individuals in group E_{ij} would all play the same (possibly mixed) strategy from S_i. The next step, then, would be to randomly play members of the populations against each other. The sub-populations that performed the best would grow, and those that did not perform well would shrink. The process of playing members of the populations randomly and refining the populations based on performance would be repeated indefinitely. Ideally the evolution would converge to some stable state for each population, which would represent a (possibly mixed) strategy best response for each agent.

A special case is the symmetric two-player game. In a symmetric game payoff matrices and actions are identical for both agents. These games can be modelled by a single population of individuals playing against each other. When the game being played is asymmetric, a different population of players must be used to simulate each agent.

Throughout this document, the EGT approach will make use of the matrix-vector formulation of games.

If $s_1, \ldots, s_n \in S$ are the pure strategies available to a player, then that player's strategy will be denoted by the column vector $\vec{x}$. The i^{th} component of $\vec{x}$ gives the probability of playing strategy s_i. Playing a pure strategy s_j is represented by the vector whose j^{th} component is 1, and all other components are 0. When the payoff for a player is specified by a payoff matrix ***A***, a player using strategy $\vec{x}$ against an opponent with strategy $\vec{y}$ will have payoff $\vec{x}^T A\vec{y}$.

There are several critical components to an EGT analysis. The natural selection process governing the evolution of populations requires a measure of fitness for different strategies, and the process itself must be carefully chosen. Before a full discussion of the evolution process, though, it is necessary to describe evolutionary stable strategies.

THE EVOLUTIONARY DYNAMICS APPROACH

In this section we will present the second concept to analyze a game in evolutionary game theory. The general question of that approach is: How will a population of individuals that repeatedly plays a certain game evolve? The answer to that question is largely determined by the conditions under which the individuals interact. First we concentrate on a very simple setting of an infinitely large population of players with two different strategies that randomly encounter each other. In order to get not to deep into theoretical analyzes we introduce the concept by applying it on our running example the Hawk-Dove-Game.

First we have to determine all quantities and their relation with each other that are necessary to describe the dynamics of the population. Since our population is infinitely large it is sufficient to keep track of the fractions of individuals that follow a certain strategy. With p_H and p_D we denote the fractions of Hawks or Doves respectively. To model a real dynamical system we have to include some kind of reproduction. The reproduction rate should be proportional to fitness of an individual, which we denote with w_H and w_D respectively, in relation to the mean fitness $\bar{w}$. With these five quantities we can now write down the equations that relate the number of individuals in the current generation with the number of individuals in the next generation:

$$p'_H = p_H \frac{w_H}{\bar{w}} \quad (3)$$

$$p'_D = p_D \frac{w_D}{\bar{w}} \quad (4)$$

This equations are called *replicator equations* and were offered by Taylor and Jonker (1978) and Zeeman (1979). The only thing that is missing is an equation that describes the fitness of an individual. To derive that we use the assumption that the individuals meet each other randomly. If we now pick an arbitrary Hawk we can conclude that a fraction of p_H of his encounters in the current generation were encounters with other Hawks, whereas a fraction of p_D where encounters with Doves. If we now use the well-known payoffs, we get

$$w_H = p_H \Delta W_{Hawk,Hawk} + p_D \Delta W_{Hawk,Dove} \quad (5)$$

as expression for the fitness of a randomly chosen hawk. The same considerations hold for the Doves and we can immediately write down the fitness term

$$w_D = p_H \Delta W_{Dove,Hawk} + p_D \Delta W_{Dove,Dove}. \quad (6)$$

The mean fitness $\bar{w}$ can finally be calculated by

$$\bar{w} = p_H w_H + p_D w_D. \quad (7)$$

To get the dynamics of the resulting system one can either perform a computer simulation or analyze the system analytically. Since our system is quite simple we do the latter.

One of the most common tasks in the analysis of a dynamical system is the detection of fixed points. A fixed point constitutes a state of the system, were it does not change any more. To find these points we look at the reproduction equations (3) and (4). Since the sum of p_H and p_D is always equal to 1, we can concentrate on one of these equations. To find the fixed point p^*_H we make the ansatz

$$p^*_H = p^*_H \frac{w_H}{\bar{w}} \quad (8)$$

and easily see the two possibilities: a trivial one with $p^*_H = 0$ and another one when $w_H = \bar{w}$. The latter implies that either $p_D = 0$ or $w_H = w_D$. Thus we have up to 3 fixed points. Fine, but what does that mean from a biological point of view? Biologically this implies that the population is stable if either one species (Hawks or Doves) became extinct or the fitness of both is the same. Since we now know the fixed points of our system we can check whether they are stable or not. It should be obvious that $p_H = 1$ is stable iff $w_H > \bar{w} > w_D$ and $p_H = 0$ is stable if

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$w_H < \bar{w} < w_D$. To decide in which case one of the three situation $(>, <, =)$ occurs we look at the equality case. Therefore we consider the fitness equations (4.3) and (4.4), plug-in the payoff values and equate them.

$$w_H = w_D \quad (9)$$

$$p_H \frac{1}{2}(V - C) + p_D = p_H 0 + p_D \frac{V}{2} \quad (10)$$

$$p_H \frac{1}{2}(V - C) + (1 - p_H)V = (1 - p_H)\frac{V}{2} \quad (11)$$

$$V - \frac{V}{2} + p_H(\frac{V}{2} - \frac{C}{2} - V + \frac{V}{2}) = 0 \quad (12)$$

$$\frac{V}{2} = p_H \frac{C}{2} \quad (13)$$

$$p_H = \frac{V}{C} \quad (14)$$

Finally we derived a clear characterization of our system, which we can directly interpret biologically: The fitness of the strategy Hawk is always larger than or equal to the fitness of the strategy dove (this can be seen in equation (11) remembering that $V, C \geq 0$). Thus the state were all individuals in a pop ulation follow the strategy Hawk is a stable fixed point, whereas the state were all individuals follow the strategy Dove is always an unstable fixed point. A third stable fixed point (coexistence) occurs if the two strategies have the same fitness. This happens if the frequency of the Hawks equals $\frac{V}{C}$. Since p_H has to be in the range of 0 and 1 this is only possible if $C > V$ (the cost of a conflict is higher than the value of the resource). Figure 8 summarizes the behavior of the dynamical system with help of a bifurcation diagram.

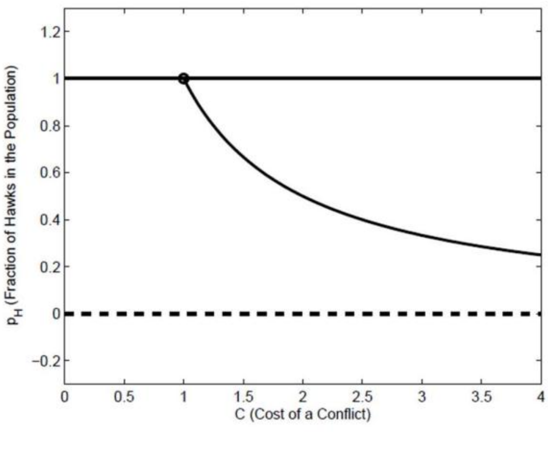

Figure 8: Bifurcation diagram for varying conflict costs and fixed resource value of the dynamical Hawk-Dove-Game system.

The fact that the strategy Hawk is an ESS and a stable fixed point in the setting of a dynamical system is no coincidence. De facto the definition of an ESS states that the system resists slight perturbations from the state were all individuals follow that strategy, which is the definition of stable fixed point.

CONCLUSION

Game theory is exciting because although the principles are simple, the applications are far-reaching. Interdependent decisions are everywhere, potentially including almost any endeavor in which self-interested agents cooperate and/or compete. Probably the most interesting games involve communication, because so many layers of strategy are possible. Game theory can be used to design credible commitments, threats, or promises, or to assess propositions and statements offered by others. Advanced concepts, such as brinkmanship and inflicting costs, can even be found at the heart of foreign policy and nuclear weapons strategies . some the most important decisions people make.

Evolutionary game theory is a maturing field; many basic theoretical issues are well understood, but many difficult questions remain. It is tempting to say that stochastic and local interaction models offer the more open terrain for further explorations. But while it is true that we know less about these models than about deterministic evolutionary dynamics, even our knowledge of the latter is limited: while dynamics on one and two dimensional state spaces, and for games satisfying a few interesting structural assumptions, are well-understood, the dynamics of behavior in the vast majority of many-strategy games are not.

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