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An Analysis upon Some Properties of Determinantal Point Processes and Zeros of Certain Gaussian Analytic Functions

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Abstract – Random configurations of points in space, also known as point processes, have been studied in mathematics, statistics and physics for many decades. In mathematics and statistics, the emphasis has been on the Poisson process, which can be thought of as a limit of picking points independently and uniformly in a large region.

Taking a different perspective, a finite collection of points in the plane can always be considered as the roots of a polynomial; in this coordinate system, taking the coefficients of the polynomial to be independent is natural. Limits of these random polynomials and their zeros are a core subject of this study; the other class consists of processes with joint intensities of Determinantal form.

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INTRODUCTION

The primary objects of study in this study are point processes, which are random variables taking values in the space of discrete subsets of a metric space, where, by a discrete set we mean a countable set with no accumulation points. Precise definitions of relevant notions will be given later. Many physical phenomena can be modeled by random discrete sets. For example, the arrival times of people in a queue, the arrangement of stars in a galaxy, energy levels of heavy nuclei of atoms etc. This calls upon probability to find point processes that can be mathematically analysed in some detail, as well as capture various qualitative properties of naturally occurring random point sets.

The single most important such process, known as the **Poisson process** has been widely studied and applied. The Poisson process is characterized by independence of the process when restricted to disjoint subsets of the underlying space. More precisely, for any collection of mutually disjoint measurable subsets of the underlying space, the numbers of points of a Poisson process that fall in these subsets are stochastically independent. The number of points that fall in A has Poisson distribution with a certain mean $\mu(A)$ depending on A. Then, it is easy to see then that μ must be a measure, and it is called the intensity measure of the Poisson process. This assumption of independence is acceptable in some examples, but naturally, not all. For instance if one looks at outbreaks of a rare disease in a province, then knowing that there is a case in a particular location makes it more likely that there are more such cases in a neighborhood of that location. On the other hand, if one looks at the distribution of like-charged particles confined by an external field (physicists call it a “one component plasma”), then the opposite is true. Knowing that a particular location holds a particle makes it unlikely for there to be any others close to it. These two examples indicate two ways of breaking the independence assumption, inducing more clumping (“positively correlated”) as in the first example or less clumping (“negatively correlated”) as in the second.

A natural question is, are there probabilistic mechanisms to generate such clumping or anti-clumping behaviour? A simple recipe that gives rise to positively correlated point processes is well-known to statisticians: First sample $X(\cdot)$, a continuous random function on the underlying space that takes values in U+, and then, sample a Poisson process whose intensity measure has density $X(\cdot)$ with respect to a fixed reference measure v on the underlying space. These kinds of processes are now called Cox processes, and it is clear why they exhibit clumping - more points fall where X is large, and if X is large at one location in space, it is large in a neighborhood. We shall encounter a particular subclass of Cox processes, known as permanental processes, one of two important classes of point processes having negative correlations that we study in this study.

This brings us to the next natural question and that is of central importance to this study. Are there interesting point processes that have less clumping

than Poisson processes? As we shall see, one natural way of getting such a process without putting in the anti-clumping property by hand, is to extract zero sets of random polynomials or analytic functions, for instance, zeros of random polynomials with stochastically independent coefficients. On the other hand it is also possible to build anti-clumping into the very definition. A particularly nice class of such processes, known as determinantal point processes, is another important object of study in this study.

We study these point processes only in the plane and give some examples on the line, that is, we restrict ourselves to random analytic functions in one variable. One can get point processes in $\mathbb{R}^{2n}$ by considering the joint zeros of n random analytic functions on $\mathbb{C}^n$, but we do not consider them in this study. Determinantal processes have no dimensional barrier, but it should be admitted that most of the determinantal processes studied have been in one and two dimensions. In contrast to Cox processes that we described earlier, determinantal point processes seem mathematically more interesting to study because, for one, they are apparently not just built out of Poisson processes.

Next we turn to the reason why these processes (zeros of random polynomials and determinantal processes) have less clustering of points than Poisson processes. Determinantal processes have this anti-clustering or repulsion built into their definition, and below we give an explanation as to why zeros of random polynomials tend to repel in general. Before going into this. All the three samples shown are portions of certain translation invariant point processes in the plane, with the same average number of points per unit area. Nevertheless, they visibly differ from each other qualitatively, in terms of the clustering they exhibit.

Now we “explain” the repulsion of points in point processes arising from zeros of random analytic functions (Of course, any point process in the plane is the zero set of a random analytic function, and hence one may wonder if we are making an empty or false claim. However, when we use the term random analytic function, we tacitly mean that we have somehow specified the distribution of coefficients and that there is a certain amount of independence therein). Consider a polynomial

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0. \qquad (1)$$

We let the coefficients be random variables and see how the (now random) roots of the polynomial are distributed. This is just a matter of change of variables, from coefficients to the roots, and the Jacobian determinant of this transformation is given by the following well known fact.

LEMMA 1 .Let $p(z) = \prod_{k=1}^{n}(z - z_k)$ have coefficients $a_k,\ 0 \le k \le n-1$ as in (1.1.1).

Then the transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(z_1, \ldots, z_n) = (a_{n-1}, \ldots, a_0)$, has Jacobian determinant $\prod_{i<j} |z_i - z_j|^2$.

PROOF. Note that we are looking for the real Jacobian determinant, which is equal to

$$\left| \det\left(\frac{\partial T(z_1, \ldots, z_n)}{\partial (z_1, \ldots, z_n)} \right) \right|^2.$$

To see this in the simplest case of one complex variable, observe that if $f = u + iv : \mathbb{C} \to \mathbb{C}$, its Jacobian determinant is

$$\det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix},$$

which is equal to $|f'|^2$, provided f is complex analytic. See Exercise 1 for the relationship between real and complex Jacobian determinants in general.

Let us write

$$T_n(k) = a_{n-k} = (-1)^k \sum_{1 \le i_1 < \ldots i_k \le n} z_{i_1} \ldots z_{i_k}.$$

$T_n(k)$ and all its partial derivatives are polynomials in z_js. Moreover, by the symmetry of $T_n(k)$ in the z_js, it follows that if $z_i = z_j$ for some $i \ne j$, then the i^{th} and j^{th} columns of $\frac{\partial T(z_1, \ldots, z_n)}{\partial (z_1, \ldots, z_n)}$ are equal, and hence the determinant vanishes. Therefore, the polynomial $\det\left(\frac{\partial T_n(k)}{\partial z_j}\right)_{1 \le j,k \le n}$ is divisible by $\prod_{i<j}(z_i - z_j)$. As the degree of $\det\left(\frac{\partial T_n(k)}{\partial z_j}\right)_{1 \le j,k \le n}$ is equal to $\sum_{k=1}^{n}(k-1) = \frac{1}{2}n(n-1)$, it must be that

$$\det\left(\frac{\partial T(z_1, \ldots, z_n)}{\partial (z_1, \ldots, z_n)} \right) = C_n \prod_{i<j}(z_i - z_j).$$

To find the constant C_n, we compute the coefficient of the monomial $\prod z_j^{j-1}$ on both sides. On the right hand side the coefficient is easily seen to be $D_n := (-1)^{n(n-1)/2} C_n$.

On the left, we begin by observing that $T_n(k) = -z_n T_{n-1}(k-1) + T_{n-1}(k)$, whence

$$\frac{\partial T_n(k)}{\partial z_j} = -z_n \frac{\partial T_{n-1}(k-1)}{\partial z_j} + \frac{\partial T_{n-1}(k)}{\partial z_j} - \delta_{jn} T_{n-1}(k-1). \qquad (2)$$

The first row in the Jacobian matrix of T has all entries equal to -1. Further, the entries in the last column (when $j = n$) are just $-T_{n-1}(k-1)$, in particular, independent of z_n. Thus when we expand $\det\left(\frac{\partial T_n(k)}{\partial z_j}\right)$:>y the first row, to get z_n^{n-1} we must take the entry in the first row and in every other row we must use the first summand in (2) to get a factor of z_n. Therefore

$$\begin{aligned} D_n &= \text{coefficient of } \prod_{j=1}^{n} z_j^{j-1} \text{ in } \det\left(\frac{\partial T_n(k)}{\partial z_j}\right)_{1\le k,j\le n} \\ &= (-1)^n \text{ coefficient of } \prod_{j=1}^{n-1} z_j^{j-1} \text{ in } \det\left(-\frac{\partial T_{n-1}(k-1)}{\partial z_j}\right)_{\substack{2\le k\le n \\ 1\le j\le n-1}} \\ &= -D_{n-1}. \end{aligned}$$

Thus $C_n = (-1)^n C_{n-1} = (-1)^{n(n+1)/2}$ because $C_1 = -1.$ Therefore the real Jacobian determinant of T is $\prod_{i<j} |z_i - z_j|^2.$

GAUSSIAN ANALYTIC FUNCTIONS

Complex Gaussian distribution-

Throughout this study, we shall encounter complex Gaussian random variables. As conventions vary, we begin by establishing our terminology. By $N(\mu,\sigma^2)$, we mean the distribution of the real-valued random variable with probability density $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Here $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ are the mean and variance respectively.

A **standard complex** Gaussian is a complex-valued random variable with probability density $\frac{1}{\pi} e^{-|z|^2}$ w.r.t the Lebesgue measure on the complex plane. Equivalently, one may define it as $X+iY$, where X and Y are i.i.d. $N(0,\frac{1}{2})$ random variables.

Let $a_k,\ 1 \le k \le n$ be i.i.d. standard complex Gaussians. Then we say that $\mathbf{a} := (a_1,\ldots,a_n)^t$ is a standard complex Gaussian vector. Then if B is a (complex) $m \times n$ matrix, $B\mathbf{a}+\mu$ is said to be an ra-dimensional complex Gaussian vector with mean p(an m x 1 vector) and covariance $\Sigma = BB^*$(an $m \times m$ matrix). We denote its distribu tion by $N_{\mathbb{C}}^m(\mu,\Sigma).$

EXERCISE 2. Let U be an nxn unitary matrix, i.e. $UU^* = I_d$,(here U^* is the conjugate transpose, and a an n-dimensional standard complex Gaussian vector. Show that $U\mathbf{a}$ is also an n-dimensional standard complex Gaussian vector, ii. Show that the mean and covariance of a complex Gaussian random vector determines its distribution.

Gaussian analytic functions-

Endow the space of analytic functions on a region $\Lambda \subset \mathbb{C}$ with the topology of uniform convergence on compact sets. This makes it a complete separable metric space which is the standard setting for doing probability theory (To see completeness, if $\{f_n\}$ is a Cauchy sequence, then f_n converges uniformly on compact sets to some continuous function f. Then Morera's theorem assures that that f must be analytic because its contour integral vanishes on any closed contour in Λ, since $\int_\gamma f = \lim_{n\to\infty} \int_\gamma f_n$ and the latter vanishes for every n by analyticity of f_n).

DEFINITION 1. Let f be a random variable on a probability space taking values in the space of analytic functions on a region $\Lambda \subset \mathbb{C}$. We say f is a Gaussian analytic function (GAF) on Λ if $(\mathbf{f}(z_1),\ldots,\mathbf{f}(z_n))$ has a mean zero complex Gaussian distribution for every $n \ge 1$ and every $z_1,\ldots,z_n \in \Lambda.$

It is easy to see the following properties of GAFs

- $\{\mathbf{f}^{(k)}\}$ are jointly Gaussian, i.e., the joint distribution of $\mathbf{f}$ and finitely many derivatives of $\mathbf{f}$ at finitely many points, $\{\mathbf{f}^{(k)}(z_j): 0 \le k \le n, 1 \le j \le m\}$, has a (mean zero) complex Gaussian distribution. (Hint: Weak limits of Gaussians are Gaussians and derivatives are limits of difference coefficients).
- For any $n \ge 1$ and any $z_1,\ldots,z_n \in \Lambda$, the $\mathbb{C}^n$-valued random vector $(\mathbf{f}(z_1),\ldots,\mathbf{f}(z_n))$ has a complex Gaussian distribution with mean zero and covariance matrix $(K(z_i,z_j))_{i,j\le n}$. By Exercise 2 it follows that the covariance kernel K determines all the finite dimensional marginal of $\mathbf{f}$ Since $\mathbf{f}$ is almost surely continuous, it follows that the distribution of $\mathbf{f}$ is determined by K.
- Analytic extensions of GAFs are GAFs.

ZEROS OF GAUSSIAN ANALYTIC FUNCTIONS

Throughout this note we shall use the following notation. Let $G \subseteq \mathbb{C}^1$ be a plane domain and $\{\psi_j(z)\}_{j=1}^N$ be a system of $N \le \infty$ analytic functions in G. By $\Psi(z)$ we denote the holomorphic curve in the euclidean space $\mathbb{C}^N$ with coordinates $\psi_j(z)$. If $N = \infty$, we assume that

$$||\Psi(z)|| = \sum_j |\psi_j(z)|^2 < \infty, \qquad z \in G, \tag{1}$$

where the series on the RHS converges locally uniformly in G.

Let ω_j be independent, complex-valued, gaussian random variables such that

$$\mathcal{E}\{\omega_j\} = 0, \quad \text{and } \mathcal{E}\{|\omega_j|^2\} = 1.$$

We identify the probability space with $\mathbb{C}^N$ equipped with the gaussian product measure $d\nu$.

A gaussian analytic function $\psi(z,\omega)$ is defined as

$$\psi(z,\omega) = \sum_j \omega_j \psi_j(z) \qquad (2)$$

If $N = \infty$, then according to a theorem by Khintchin and Kolmogorov, the series converges locally uniformly in G and almost surely in ω, and hence defines an analytic function in G.

Let n_ω be a counting measure of zeros (according to their multiplicities) of the function $\psi(z,\omega)$. Here we shall be concerned with three general results on the random measure n_ω. The first one is a formula for the average $\mathcal{E}\{n_\omega\}$ which is due to Edelman and Kostlan. The second, close to Calabi's Rigidity Theorem [4], loosely speaking says that the average measure $\mathcal{E}\{n_\omega\}$ "almost determines" the analytic functions $\psi_j(z)$. The third result, which is due to Offord, is an exponential decrease of "tail probabilities" of an analytic function having an excess or deficiency of zeros in a given region. An important feature of these three results is that they do not need any assumptions about analytic functions $\psi_j(z)$ and "dimension" N. We shall not touch on the more delicate statistics of the local correlation functions, which was recently of some interest in mathematical physics (see references at the end of this note). By C and c we denote various positive numerical constants which may vary from line to line.

DETERMINANTAL POINT PROCESSES

In this chapter we move away from zeros of random analytic functions and study a different class of point processes known as determinantal point processes. These arise surprisingly often, in random matrix theory, combinatory and physics. Many examples were already known before Macchi introduced the general notion in 1975. To motivate the definition, we remind the reader that in quantum mechanics, a physical quantity, say the position of an electron, is represented by a complex valued function (the wave function) ψ such that $\int |\psi|^2 = 1$.

Then $|\psi|^2$ gives the probability density function of the position. Now consider individual wave functions $\psi_1, \ldots, \psi_n$ on Λ. The most obvious way to construct an rc-particle wave function out of the ψ_i s is to Consider

$$(\psi_1 \otimes \ldots \otimes \psi_n)(x_1, \ldots, x_n) = \prod_{i=1}^{n} \psi_i(x_i),$$

which is tantamount to making the individual positions be independent random variables. This does not capture the physical reality, for electrons repel, and moreover the particles are indistinguishable. For this reason, physicists symmetrize or anti- symmetrize the wave-function $\psi_1 \otimes \ldots \otimes \psi_n$, either of which leads to a summarization of the probability density. We shall consider anti-summarization here. Symmetrizing $\psi_1 \otimes \ldots \otimes \psi_n$ would lead to permanental point processes.

For particles with repulsion ("fermions"), one should anti-symmetrize and this yields the wave function

$$\frac{1}{\sqrt{n!}} \sum_{\pi \in \mathbb{S}_n} \operatorname{sgn}(\pi) \prod_{i=1}^{n} \psi_{\pi_i}(x_i) = \frac{1}{\sqrt{n!}} \det\left(\psi_j(x_i)\right)_{i,j \le n}.$$

If $\{\psi_i\}$ is orthonormal, then the absolute square of this wave function is a probability density, for, integrating $\prod_{i=1}^{n} \psi_{\pi_i}(x_i)$ against $\prod_{i=1}^{n} \overline{\psi}_{\sigma_i}(x_i)$ gives zero unless $\pi = \sigma$. Thus we get the probability density on Λ^n $\frac{1}{n!} \det\left(\psi_j(x_i)\right) \det\left(\overline{\psi}_i(x_j)\right) = \frac{1}{n!} \det\left(\mathbb{K}(x_i, x_j)\right)_{i,j \le n}$, where Note that the probability density vanishes whenever $x_i = x_j$ for some $i \neq j$. $\mathbb{K}(x,y) = \sum_{i=1}^{n} \psi_i(x) \overline{\psi}_i(y)$. which indicates that the points tend to "repel".

There is one more step required. If we want to define analogous point processes with infinitely many points, or even to effectively study local properties of the finite ones, we need to have the joint intensities. Here a fortuitous simplification occurs which is at the very heart of the virtues of a determinantal point process.

REFERENCES

A. C. Offord (1967). The distribution of zeros of power series whose coefficients are independent random variables, Indian J. Math. 9, pp. 175–196.

Alon Nishry. Asymptotics of the hole probability for zeros of random entire functions.

Edelman and E. Kostlan (1995). How many zeros of a random polynomial are real? Bull. Amer. Math. Soc. 32, pp. 1–37.

Itai Benjamini, Russell Lyons, Yuval Peres, and Oded Schramm (2001). Uniform spanning forests, Ann. Probab. 29, no. 1, pp. 1–65. MR MR1825141 (2003a:60015)

J. Daley and D. Vere-Jones (2004). An introduction to the theory of point processes. Vol. I, second ed., Probability and its Applications (New York), Springer- Verlag, New York, 2003, Elementary theory and methods. MR MR1950431 (2004c:60001)

Kurt Johansson (2002). Non-intersecting paths, random tilings and random matrices, Probab. Theory Related Fields 123, no. 2, pp. 225–280. MR MR1900323 (2003h:15035)

P. Bleher, B. Shiffman, and S. Zelditch (2009). Poincar´e-Lelong approach to universality

and scaling of correlations between zeros, preprint math-ph/9903012.

P. J. Forrester and G. Honner (1999). Exact statistical properties of the zeros of complex random polynomials, J. Phys. A 32, no. 16, pp. 2961–2981. MR MR1690355 (2000h:82047)

P. Leboeuf and P. Shukla (2006). Universal fluctuations of zeros of chaotic wavefunctions, J. Phys. A 29, pp. 4827–4835.

Persi Diaconis (2003). Patterns in eigenvalues: the 70th Josiah Willard Gibbs lecture, Bull. Amer. Math. Soc. (N.S.) 40, no. 2, pp. 155–178 (electronic). MR MR1962294 (2004d:15017)

Persi Diaconis and Steven N. Evans (2001). Linear functionals of eigenvalues of random matrices, Trans. Amer. Math. Soc. 353, no. 7, pp. 2615–2633. MR 2002d:60003

Richard Kenyon (2008). Height fluctuations in the honeycomb dimer model, Comm. Math. Phys. 281, no. 3, pp. 675–709. MR MR2415464

S. Nonnenmacher and A. Voros (2008). Chaoticeigenfunc tions in phase state, J. Statist. Phys. 92, pp. 431–518.

Shiffman and S. Zelditch (2009). Distribution of zeros of random and quantum chaotic sections of positive line bundles, Comm. Math. Phys. 200, pp. 661–684.

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