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Eigensystems

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Sometimes, the effect of multiplying a vector by a matrix is to produce simply a scalar multiple of that vector.

For example,

$$\begin{pmatrix} 3 & 2 \\ -4 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We often want to know to what extent this phenomenon occurs for a given matrix. That is we ask for all instances of the equation Ax=αx, where A is given but we know neither the vector x nor the scalar α. Because there are two unknown on the right multiplied together, this is not a linear problem.

If α were fixed, and we only wanted to find all the appropriate x-vectors, if would be a standard linear problem. Even easier is the ease when x is known, and we only want to determine α.

Each of these is linear problem. But it is nonlinear if x and α are both unknown. Therefore, we must expect to encounter some obstacles in solving this problem.

EIGENVECTORS AND EIGENVALUES

Definition:-

Let A be any square matrix, real or complex. A number α is an eigenvalue of A if the equation Ax = α x is true for some non-zero vector x. (Here α is allowed to be a real or complex number.) The vector x is an eigenvector associated with the eigenvalue α. The eigenvector may also be complex.

Arithmetic with complex numbers cannot be avoided in the subject of eigenvalues. That is why the preceding definition explicitly allows for complex numbers to enter our calculations.

In matrix theory, some terminology has changed over time. For example, eigenvalues have been known as characteristic values, latent roots and eigenvalues.

As recently as 1949, the eminent mathematician D.E little wood was calling an eigenvalues a “latent root” and eigenvector a “pole” and the inverse of a matrix its “reciprocal”.

The German word “Eigen” has several different English translations, such as proper, peculiar and strange.

Summary and Examples:-

The key equation is Ax = αx.

Most vectors x will not satisfy such an equation. They change direction when multiplied by A, so that Ax is not a multiple of x.

This means that only certain special numbers α are eigenvalues and only certain special vectors x are eigenvectors. We can watch the behavior of each eigenvector, and then combine there “normal modes” to find the solution.

Example 1 – The eigenvalues of a projection matrix are 1 or 0 !

P = $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has α_l = 1 with x_1 = $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

x_2=0 with x_2= $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

we have α = 1 when x projects to itself , and α =0 when x projects to the zero vector.

The column space of P is filled with eigenvectors, and so is the null space. If those spaces have dimension r and n-r, then α=1 is repeated r times and α=0 is repeated n-r times (always n α’s).

0P = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ has $\alpha = 1,1,0,0$

There is nothing exceptional about α=0.

Like every other number, zero might be an eigenvalue and it might not. If it is, then its eigenvector satisfy Ax = 0x, Thus x is in the null space of A. A zero eigenvalue signals that A is singular (not invertible);

Its determinant is zero.

Invertible matrices have all α≠0.

Example-2:- The eigenvalues are on the main diagonal when A is triangular.

det (A- αI) = $\begin{vmatrix} 1-\alpha & 1 & 1 \\ 0 & 0-\alpha & 0 \\ 0 & 0 & 0-\alpha \end{vmatrix}$ = (1- α) (2- α) $(4-\alpha)$

= (1- α) $(2-\alpha)(4-\alpha)$

The determinant is just the product of the diagonal entries. It is zero if α=1, α=2, α=4.

The Eigen values were sitting along the main diagonal.

Eigen values for matrices:-

The operation of multiplying vectors in F^n by a matrix A € F^{n*n} is a linear transformation of F^n into itself.

- A point α € F is said to be an eigenvalue of A € F^{n*n} if there exists a nonzero vector u € F^n such that Au = αu i.e. $N_{(A-\alpha In)} \neq \neq \{0\}$.

Every non-zero vector u € $N_{(A-\alpha In)}$ is said to be an eigenvector of the matrix A € F^{n*n} corresponding to the eigenvalue.

- A non-zero vector u € F^n is said to be generalized eigenvector of the matrix A € F^{n*n} corresponding to the eigenvalue α € F if u € $N_{(A-\alpha In)})^n$

- A vector u € F^n is said to be a generalized eigenvector of order K of the matrix A € F^{n*n} corresponding to the eigenvalue α€ F if $(A-\alpha I_n)^k\, u = 0$, but $(A-\alpha I_n)^{k-1}$ u≠0. In this instance α€F the set of vectors $u_j = (A-\alpha I_n)^{k-j}\, u$ for j=1,…,k is said to form a Jordan chain of length K, they satisfy the following chain of equalities

(A- αI_n) u_1 = 0

αI_n) u_2 = u_1

(A – I_n) u_k = u_k -1

Thm:-

If $V_1, V_2,$ ………… $V_r,$ are eigenvectors that correspond to distinct evgenvalues $\alpha_1, \alpha_2,$ ………… $\alpha_r,$of nxn matrix A, then the set $\{V_1, V_2,$ ………… $V_r,\}$ is linearly independent.

Proof:-

Suppose $\{V_1, V_2,$ ………… $V_r,\}$ is linearly dependent. Since V_1 is on zero, then one of the vectors in the set is linear combination of the preceding vectors. Let q be the least index such that Vq+1 is a linear combination of the preceding vectors. Then there exist scalars C_1………….C_2 such that

$C_1V_1 + C_2V_2$……… $+ C_qV_q = V_q + 1$

$C_1AV_1 +$ ……………… $C_qAV_q = AV_{q+1}$

$C_1\alpha v_1 +$ ……………… $C_q\alpha_qV_q = \alpha_{q+1}V_{q+1}$

Multiplying both sides of (1) by α_{q+1} and subtracting the result form (2), we get.

Since V_1………….V_2 is linearly independent, then weights in (3) are all zero. But none of the factors $\alpha_i - \alpha_{q+1}$ are zero, because the eigenvalues are distinct. Hence $C_{1=0}$ For i=1,…….q.

But then (1) says that $V_{q+1=0}$, which is impossible. Hence V_1………….V_r cannot be linearly dependent are therefore must be linearly independent.

Using Determinants in Finding Eigenvalves:-

If α is an eigenvalue of A, then the equation Ax=αx has a non-trivial solution. Consequently, The equation (A- αI)x=0 has a non-trivial solution and the matrix A-αI is non-invertible. Hence we have det (A- α I)=0.

Theorem:-

A scalar α is an eigenvalue of a matrix A if and only if Det (A- α I) =0.

The effect of this theorem is to turn the intractable nonlinear problem into a problem involving only α. conceptually this is very important, although finding the correct values of α may still be numerically challenging.

The equation Det (A- α I) =0 is called characteristic equation of A. It is the equation from which we can compute the eigenvalues of A. the function α Det (A- α I) is the characteristic polynomial of A.

Eigen systems without Determinants:-

Many professional mathematicians advocate a separation of the subject of eigenvalues form determinant theory. This view of the subject is backed by the knowledge gained form numerical analysis, where determinants are almost near used.

Theorem:-

Every linear operator defined on and taking values in a finite-dimensional vector space has at least one eigenvalue.

Proof:-

Let L be such an operator, and let u be a nonzero vector in the domain of L. The sequence {u, l (4), L^2 (4)} cannot be linearly independent because the space involved here is finite dimensional.

If each factor L-r_iI were injective, p(L)u would not be o, since we started with the nonzero vector u. Thus, for some value of I, L-r_iI is not injective and we have a nonzero. Solution to the equation L(x) – rix=o

Eigen spaces:-

If l be the linear transformations

L: $R^3 \rightarrow R^3$ given by L $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ = $\begin{pmatrix} 2x+y-z \\ x=2y-z \\ -x-y+2z \end{pmatrix}$

In the standard basis the matrix M representing L has columns lie for each I, so;

$\begin{pmatrix} x \\ y \\ l \end{pmatrix}_{\underset{\rightarrow}{L}} \quad \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

From here we get l has eigenvalues $\alpha_{i=1}$(with multiplicy2) and $\alpha_{2=4}$

Here we found two eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ for L, both with eigenvalue 1.

Notec that $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is also eigenuctor of L with eigenvalue 1. Infact any linear combinatior r $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ of these two eigenvectors.

will be another Eigen vector with same Eigen value. More generally, let $\{V_1, V_2 \ldots\ldots\ldots\}$ be eigenvector with the same Eigen value of some linear transformation L.

A linear combination of the V_i combination is given by

$(C^1V_2 + C^2V_2 + \ldots\ldots\ldots)$ for some constant C^1, C^2, ……………, Them L $(C^1V_1 + C^2V_2 + \ldots\ldots)$ = $C^1LV_1 + C^2LV_2 +$ ….. by linearity of L

So eiury linear combination of the …… is an eigenvector of L with the same eigenvalue α.

In simple terms, any sum of eigenvector is again an eigenvector if they share the same eigenvalue. The space of all vectors with eigenvalue α is called an Eigen space. It is Infact, a vector space confined within the larger vector space V. it contains O_{vi} since LO_v = O_v = αOV, and is closed under addition and scalar multiplication.

Dynamical Systems and Eigenvectors:-

Consider a 7 physical system whose state at any given time it is described by some quantities x_i(t), x_2(t) ………… $x_n(t)$. we can represent the quantities x_i(t), x_2(t) ….. $x_n(t)$. by the state vector.

Suppose that the state of the system at time i+l is determined by the state at time t and that the transformation of the system form time t to time t+l is linear, represented by nxn matrix A.

$x^{\rightarrow}$ (t) = $AX^{\rightarrow}$(t)

Then $x^{\rightarrow}$ (t) = $A^{t\rightarrow}x^0$

Such a system is called a discrete linear dynamical system. (Discrete indicates that we model the change of the system form time t to time t+l, rather than modeling the continuous rate if change, which is descried by differential equations.)

For an initial state, it is often our goal to find closed formulas for x_i (t), x_2 (t), ………. x_n (t) (i.e., formulas expressing x_i (t) as a function of t alone, as opposed to a recursive formula, for example, which would merely express x_i (t+i), x_2 (t), …………… , x_n (t).

REFERENCES

Linear Algebra (Theory and Application) by "word chancy, David kincaid, "Chapter – 6"

Linear Algebra and its applications III_{rd} edition by "David C Lay". Chapter – 5

Linear Algebra and its Applications by "Gilbert Strang". Chapter – 5

Linear Algebra by "David chernery, Tom Denton, Rohit Thomas and Andrew waldron." Chapter - 12

Linear Algebra in Action. by "Harry Bym" (Indian Edition) Chapter – 4

Linear Algebra with Application III_{rd} edition by "Otto Bretscher" Chapter - 7

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