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An Analysis upon Various Numerical Methods of Dispersive Partial Differential Equations

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Abstract – At the point when postured on a periodic domain in one space variable, linear dispersive advancement equations with integral polynomial dispersion relations show strikingly distinctive practices relying on whether the time is sound or silly in respect to the length of the interim, subsequently delivering the Talbot impact of dispersive quantization and racialization.

In the present study, we execute Reduced Differential Transform Method to around understand the nonlinear dispersive $K(m, n)$ sort equations. This method is an option way to deal with defeat the negative mark of complex calculation of differential transform method, equipped for decreasing the measure of calculation and effortlessly conquers the trouble of the perturbation technique or Adomian polynomials. To outline the utilization of this method, the two unique cases, $K(2, 2)$ and $K(3, 3)$ are talked about.

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INTRODUCTION

Nonlinear dispersive and wave equations are essential models to numerous territories of physics and building like plasma physics, nonlinear optics, Bose-Einstein condensates, water waves, and general relativity. Illustrations incorporate the nonlinear Schrondinger, wave, Klein-Gordon, water wave, and Einstein's equations of general relativity. This field of PDE has seen a blast in movement in the previous twenty, halfway on account of a few effective cross-fertilizations with different zones of science; chiefly symphonious analysis, dynamical systems, and likelihood. It additionally keeps on being a standout amongst the most dynamic territories of exploration, rich with problems and open to numerous fascinating bearings.

The course is proposed as a prologue to nonlinear dispersive PDE, with a goal of uncovering some open inquiries and bearings that are fruitful regions for future exploration. In the course of recent decades, a broad collection of studies have added to the mathematical theories of different classes of dispersive equations; and the logical results, similar to local and global well-posedness hypothesis, presence and uniqueness of stationary states et cetera, are rich and unlimited in the writing (see, e.g., some late monographs on this point). In parallel with the expository studies, a surge of endeavors have been dedicated to the numerics of these equations, which is a subject of extraordinary interests from the perspective of solid true applications to physics and different sciences. Despite the fact that the numerical estimate of solutions of differential equations is a customary theme in numerical analysis, has a long history of improvement and has never halted, it stays as the pulsating heart in this field to propose more modern numerical methods for dispersive equations.

The most basic asymptotic equation is likely the nonlinear Schrodinger equation, which depicts wave trains or frequency envelopes near a given frequency, and their self cooperations. The Korteweg-de-Vries equation ordinarily happens as first nonlinear asymptotic equation when the earlier linear asymptotic equation is the wave equation. It is one of the astonishing realities that numerous nonexclusive asymptotic equations are integrable as in there are numerous formulae for particular solutions.

In the mid 1990's, Michael Berry, found that the time advancement of unpleasant starting information ON periodic domains through the free space linear Schrodinger equation displays fundamentally diverse conduct contingent on whether the slipped by time is a sound or nonsensical different of the length of the space interim. Specifically, given a stage capacity as starting conditions, one finds that, at sane times, the arrangement is piecewise steady, yet discontinuous, while at nonsensical times it is a continuous however no place separate fractal-like functions. All the more for the most part, when beginning with more broad introductory information, the arrangement profile at discerning times is a linear mix of limitedly numerous interprets of the underlying information, which explains the presence of piecewise steady profiles acquired when beginning with a stage capacity. Berry named this striking marvel the Talbot impact, after a fascinating optical trial initially performed by the innovator of the photographic negative.

As indicated by (Olver, P.J. Furthermore, Oskolkov, K.I.), it was demonstrated that the same Talbot impact of dispersive quantization and fractalization shows up all in all periodic linear dispersive equations whose dispersion connection is a various of a polynomial with number coefficients (an "integral polynomial"), the prototypical case being the linearized Korteweg-deVries equation. Subsequently, it was numerically observed, that the effect persists for more general dispersion relations which are asymptotically polynomial: $\omega(k) \sim c\,k^n$ for large wave numbers $k \gg 0$, where $c \in \mathbb{R}$ and $2 \le n \in \mathbb{N}$. In any case, equations having other huge wave dispersive asvmptotics show a wide assortment of captivating and up 'til now ineffectively comprehended practices, incorporating huge scale oscillations with step by step collecting waviness, dispersive oscillations prompting a slightly fractal wave structure superimposed over a gradually swaying sea, gradually changing voyaging waves, oscillatory waves that interface and in the end get to be fractal, and completely fractal quantized conduct. Up 'til now, aside from the integral polynomially dispersive case, every one of these outcomes depend on numerical perceptions, and, regardless of being basic linear partial differential equations, thorough articulations and verifications have all the earmarks of being extremely troublesome. The concentrate likewise showed some preliminary numerical calculations that firmly demonstrate that the Talbot impact of dispersive quantization and fractalization holds on into the nonlinear administration for both integral and non-integrable development equations whose linear part has an integral polynomial dispersion connection.

The objective of the present study is to proceed with our investigations of the impact of periodicity on harsh starting information for nonlinear advancement equations with regards to two critical illustrations: the nonlinear Schrodinger (nlS) and Korteweg-deVries (KdV) equations, having, individually, rudimentary second and third request monomial dispersion. Our basic numerical tool is the administrator part method, which serves to highlight the transaction between the practices impelled by the linear and nonlinear parts of the equation. Prior thorough results concerning the administrator part method for the Korteweg-deVries, summed up Korteweg-deVries, and nonlinear Schrodinger equations can be found in different studies (Holden, H. also, Lubich, C). We likewise allude the peruser and the references in that for an examination of alternative numerical plans and meeting thereof for L2 introductory information on the genuine line.

Since a preparatory adaptation of this study seemed on the web, Erdogan and Tzirakis, have now demonstrated the Talbot impact for the integrable nonlinear Schrodinger equation, demonstrating that at judicious times the arrangement is quantized, while at irra¬tional times it is a continuous, no place separate capacity with fractal profile, in this way affirming our numerical examinations. Thoroughly setting up such observed impacts in the third request Korteweg-deVries equation, and also nonlinear Schrodinger equations with more broad nonlinearities stays open problems.

DISPERSIVE PARTIAL DIFFERENTIAL EQUATIONS

A partial differential equation (PDE) is called dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. As an example consider $iu_t + u_{xx} = 0$. If we try a simple wave of the form $u(x,t) = Ae^{i(kx-\omega t)}$, we see that it satisfies the equation if and only if $\omega = k^2$. This is called the dispersive relation and shows that the frequency is a real valued function of the wave number. If we denote the phase velocity by $v = \frac{\omega}{k}$ we can write the solution as $u(x,t) = Ae^{ik(x-v(k)t)}$ and notice that the wave travels with velocity k. Thus the wave propagates in such a way that large wave numbers travel faster than smaller ones. (Trying a wave solution of the same form to the heat equation $u_t - u_{xx} = 0$, we obtain that the lj is complexd valued and the wave solution decays exponential in time. On the other hand the transport equation $u_t - u_x = 0$ and the one dimensional wave equation $u_{tt} = u_{xx}$ are traveling waves with constant velocity.)

If we add nonlinear effccts and study $iu_t + u_{xx} = f(u)$, we will see that even the existence of solutions over small times requires delicate techniques.

Going back to the linear equation, consider $u_0(x) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx}dk$. For each fixed k the wave solution becomes $u(x,t) = \hat{u}_0(k)e^{ik(x-kt)} = \hat{u}_0(k)e^{ikx}e^{-ik^2t}$. Summing over k (integrating) we obtain the solution to our problem

$$u(x,t) = \int_{\mathbb{R}} \hat{u}_0(k)e^{ikx-ik^2t}dk.$$

Since $|\hat{u}(k,t)| = |\hat{u}_0(k)|$ we have that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Hence the preservation of the L^2 standard (mass protection or total probability) and the way that high frequencies travel quicker, prompts the conclusion that not just the arrangement will scatter into independent waves yet that its plentifulness will rot after some time. This is not any longer the situation for solutions over minimized domains. The dispersion is constrained and for the nonlinear dispersive problems we see a relocation from low to high frequencies. This fact is captured by zooming more closely in the Sobolev norm

$$\|u\|_{H^s} = \sqrt{\int |\hat{u}(k)|^2(1+|k|)^{2s}dk}$$

and observing that it actually grows over time. To analyze further the properties of dispersive PDEs and outline some recent developments we start with a concrete example.

NONLINEAR DISPERSIVE $K(m, n)$ EQUATIONS BY RDT METHOD

Seeking the singular solutions of nonlinear equations has unequivocal part in mathematical physics. There are numerous nonlinear equations material in building, liquid mechanics, science, hydrodynamics and physics (for instance plasma physics, strong state physics, liquid mechanics, for example, Korteweg-de Wries (KdV) equation, mKdV equation, RLW equation, Sine-Gordon equation, Boussinesq equation, Burgers equation, and so on. Firstly Wadati created KdV arrangement and the mKdV arrangement. Here, we say a basic type of the surely understood KdV equation.

$$u_t - auu_x + u_{xxx} = 0. \tag{1}$$

The dispersion term u_{xxx} in the equation (1) makes the wave structure spread. Solitons has been concentrated on by numerous numerical and systematic methods, for example, Adomian deterioration method, homotopy perturbation method, variational methods, exp-capacity method, summed up helper equation method, Hirota's bilinear method, homogeneous equalization method, reverse disseminating method, sine-cosine method, differential transform method , Backlund transformation, tanh-coth method and limited contrast method.

Compactons can portrayed as solitons with limited wave length or solitons that don't have exponential tails. We can say the widths of the compactons don't rely on upon the plentifulness and they can be portrayed by the nonattendance of infinite wings.

In this study we will apply the semi-utilitarian or diminished differential transform method (RDTM) to settle the nonlinear dispersive equation, which is a compacton, called summed up KDV equation

$$u_t \pm a(u^m)_x + (u^n)_{xxx} = 0, \qquad m,n \geq 1 \tag{2}$$

firstly introduced by Rosenau and Hyman. For $K(2,2)$ and $K(3,3)$, numerical values are obtained by the RDTM and compared with the exact solution.

The basic definitions of reduced differential transform method are introduced as follows:

Definition-

If function $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the domain of interest, then let

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}, \tag{3}$$

where the t-dimensional spectrum function $U_k(x)$ is the transformed function. In this study, the lowercase $u(x,t)$ repre sent the original function while the uppercase $U_k(x)$ stand for the transformed function.

Definition-

The differential inverse transform of $U_k(x)$ is defined as follows:

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^k. \tag{4}$$

Then combining equation (3) and (4) we write

$$u(x,t) = \sum_{k=0}^{n} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0} t^k \tag{5}$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion.

For the purpose of illustration of the methodology to the proposed method, we write the nonlinear dispersive K(m, n) equation in the standard operator form

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x,t) \tag{6}$$

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with initial condition

$$u(x,0) = f(x) \tag{7}$$

Where $L = \frac{\partial}{\partial t}$ is a linear operator, $N(u(x,t)) = a(u^m)_x + (u^n)_{xxx}$ is a nonlinear term, $R(u(x,t))$ is remaining linear term and $g(x,t)$ is an inhomogeneous term.

According to the RDTM and Table 1, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x) = G_k(x) - R(U_k(x)) - N(U_k(x)) \tag{8}$$

where $R(U_k(x))$, $N(U_k(x))$ and $G_k(x)$ are the transformations of the functions $R(u(x,t))$, $N(u(x,t))$ and $g(x,t)$ respectively. We can write first few nonlinear terms as

$$N_0 = a\left(\frac{\partial}{\partial x}U_0^m(x)\right) + \left(\frac{\partial^3}{\partial x^3}U_0^n(x)\right),$$

$$N_1 = a\left(\frac{\partial}{\partial x}mU_0^{m-1}(x)U_1(x)\right) + \left(\frac{\partial^3}{\partial x^3}nU_0^{n-1}(x)U_1(x)\right)$$

$$N_2 = a\left(\frac{\partial}{\partial x}\left(m(m-1)U_0^{m-2}(x)U_1(x) + mU_0^{m-1}(x)U_2(x)\right)\right) +$$

$$\left(\frac{\partial^3}{\partial x^3}\left(n(n-1)U_0^{n-2}(x)U_1(x) + nU_0^{n-1}(x)U_2(x)\right)\right)$$

It is clear that $R(U_k(x)) = 0$ and $G_k(x) = 0$ at this equation. From the initial condition (7), we write

$$U_0(x) = f(x), \tag{9}$$

Substituting (9) into (8) and after recursive calculations, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $\{U_k(x)\}_{k=0}^{\infty}$ gives an approximate solution as,

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k \tag{10}$$

where n is the order of the approximation. Therefore, the exact solution of problem is given by

$$u(x,t) = \lim_{n\to\infty} \tilde{u}_n(x,t). \tag{11}$$

Error of the method can written as

$$\Re_{n+1}(x,t) = |u(x,t) - \tilde{u}_n(x,t)| = \sum_{k=n+1}^{\infty} U_k(x)t^k \tag{12}$$

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U(k-n)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$	$W_k(x) = (k+1)...(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!}U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x}u(x,t)$	$W_k(x) = \frac{\partial}{\partial x}U_k(x)$
$Nu(x,t)$	*Maple Code for Nonlinear Function* restart; NF:=Nu(x,t):#Nonlinear Function m:=5: # Order u[t]:=sum(u[b]*t^b,b=0..m): NF[t]:=subs(Nu(x,t)=u[t],NF): s:=expand(NF[t],t): dt:=unapply(s,t): for i from 0 to m do n[i]:=((D@@i)(dt)(0)/i!): print(N[i],n[i]); #Transform Function od:

Table 1: Reduced differential transformation

NONLINEAR DISPERSIVE EQUATIONS ON MODULATION SPACES

The theory of nonlinear dispersive equations (local and global presence, consistency, disseminating theory) is unfathomable and has been concentrated broadly by numerous creators. Exclusively, the techniques grew so far confine to Cauchy problems with introductory information in a Sobolev space, basically due to the pivotal pretended by the Fourier transform in the analysis of partial differential administrators. For an example of results and a pleasant prologue to the field, we allude the peruser to Tao's monograph and the references in that.

In this note, we concentrate on the Cauchy problem for the nonlinear Schrodinger equation (NLS), the nonlinear wave equation (NLW), and the nonlinear Klein-Gordon equation (NLKG) in the domain of modulation spaces. As a rule, a Cauchy information in a modulation space is rougher than any given one in a fragmentary Bessel potential space and this low-consistency is alluring as a rule. Modulation spaces were presented by Feichtinger in the 80s and have affirmed themselves of late as the "right" spaces in time-frequency analysis. Besides, they give a brilliant substitute in evaluations that are known not on Lebesgue spaces. This is not so much amazing, on the off chance that we consider their similarity with Besov spaces, since modulation spaces emerge basically supplanting expansion by modulation.

The equations that we will investigate are:

$$(NLS)\ i\frac{\partial u}{\partial t} + \Delta_x u + f(u) = 0,\ u(x,0) = u_0(x), \tag{13}$$

$$(NLW)\ \frac{\partial^2 u}{\partial t^2} - \Delta_x u + f(u) = 0,\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad (14)$$

$$(NLKG)\ \frac{\partial^2 u}{\partial t^2} + (I - \Delta_x)u + f(u) = 0,$$

$$u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad (15)$$

where $u(x,t)$ is a complex valued function on $\mathbb{R}^d \times \mathbb{R}$, $f(u)$ (the nonlinearity) is some scalar function of u, and u_0, u_1 are complex valued functions on $\mathbb{R}^d$ The nonlinearities considered in this study have the generic form

$$f(u) = g(|u|^2)u, \quad (16)$$

where $g \in \mathbf{A}_+(\mathbb{C})$; here, we denoted by $\mathbf{A}_+(\mathbb{C})$ the set of entire functions $g(z)$ with expansions of the form

$$g(z) = \sum_{k=1}^{\infty} c_k z^k, c_k \geq 0$$

As important special cases, we highlight nonlinear it ies that are either power-like

$$p_k(u) = \lambda |u|^{2k} u, k \in \mathbb{N}, \lambda \in \mathbb{R}, \quad (17)$$

or exponential-like

$$e_\rho(u) = \lambda(e^{\rho|u|^2} - 1)u, \lambda, \rho \in \mathbb{R}. \quad (18)$$

The nonlinearities (16) considered have the upside of being smooth. The relating equations having power-like nonlinearities pk are infrequently alluded to as arithmetical nonlinear (Schrodinger, wave, Klein-Gordon) equations. The indication of the coefficient λ decides the defocusing, missing, or centering character of the nonlinearity, at the same time, as we should see, this character will assume no part in our analysis on modulation spaces.

The classical definition of (weighted) modulation spaces that will be used throughout this work is based on the notion of short-time Fourier transform (STFT). For $z = (x, \omega) \in \mathbb{R}^{2d}$, we let M_ω and T_x denote the operators of modulation and translation, and $\pi(z) = M_\omega T_x$ the general time-frequency shift. Then, the STFT of / with respect to a window g is

$$V_g f(z) = \langle f, \pi(z)g \rangle$$

Modulation spaces provide an effective way to measure the time-frequency concentration of a distribution through size and integrability conditions on its STFT. For $s, t \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, we define the weighted modulation space $\mathcal{M}^{p,q}_{t,s}(\mathbb{R}^d)$ to be the Banach space of all tempered distributions f such that, for a nonzero smooth rapidly decreasing function $g \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\|f\|_{\mathcal{M}^{p,q}_{t,s}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x,\omega)|^p \langle x \rangle^{tp}\, dx \right)^{q/p} \langle \omega \rangle^{qs}\, d\omega \right)^{1/q} < \infty$$

Here, we use the notation

$$\langle x \rangle = (1 + |x|^2)^{1/2}$$

This definition is independent of the choice of the window, in the sense that different window functions yield equivalent modulation-space norms. When both s = t = 0, we will simply write $\mathcal{M}^{p,q} = \mathcal{M}^{p,q}_{0,0}$. It is well-known that the dual of a modulation space is also a modulation space, $(\mathcal{M}^{p,q}_{s,t})' = \mathcal{M}^{p',q'}_{-s,-t}$, where p', q' denote the dual exponents of p and q, respectively. The definition above can be appropriately extended to exponents $0 < p, q \leq \infty$ as in the works of Kobayashi. More specifically, let $\beta > 0$ and $\chi \in \mathcal{S}$ be such that supp $\hat{\chi} \subset \{|\xi| \leq 1\}$ and $\sum_{k \in \mathbb{Z}^d} \hat{\chi}(\xi - \beta k) = 1, \forall \xi \in \mathbb{R}^d$.

For $0 < p, q \leq \infty$ and $s > 0$, the modulation space $\mathcal{M}^{p,q}_{0,s}$ is the set of all tempered distributions / such that

$$\left(\sum_{k \in \mathbb{Z}^d} \left(\int_{\mathbb{R}^d} |f * (M_{\beta k}\chi)(x)|^p\, dx \right)^{\frac{q}{p}} \langle \beta k \rangle^{sq} \right)^{\frac{1}{q}} < \infty \quad (19)$$

When, $1 \leq p, q \leq \infty$ this is an equivalent norm on $\mathcal{M}^{p,q}_{0,s}$, but when $0 < p, q < 1$ this is just a quasi-norm. We refer to for more details. For another definition of the modulation spaces for all $0 < p, q \leq \infty$ we refer to. For a discussion of the cases when p and/or q = 0. These extensions of modulation spaces have recently been rediscovered and many of their known properties reproved via different methods by Baoxiang et all 1. There exist several embedding results between Lebesgue, Sobolev, or Besov spaces and modulation spaces. We note, in particular, that the Sobolev space H^2_s coincides with $\mathcal{M}^{2,2}_{0,s}$. For further properties and uses of modulation spaces, the interested reader is referred to Grochenig's book .

The objective of this note is twofold: to enhance some late consequences of Baoxiang, Lifeng and Boling on the local well-posedness of nonlinearequations expressed above, by permitting the Cauchy information to lie in any modulation space $\mathcal{M}^{p,1}_{0,s}, p \geq 1, s \geq 0$, and to improve the methods of verification by utilizing entrenched tools from time-frequency analysis. In a perfect world, one might want to adjust these methods to manage global well-posedness also. We plan to address these issues in a future work.

For the remainder of this section, we assume that $d \geq 1, k \in \mathbb{N}, 1 \leq p \leq \infty$, and $s \geq 0$ are given. Our main results are the following.

Theorem 1. Assume that $u_0 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$, and the nonlinearity f has the form (1.16).

Then, there exists $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}})$ such that (1) has a unique solution $u \in C([0,T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$

Moreover, if $T^* < \infty$, then $\limsup_{t \to T^*} \|u(\cdot,t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$

Theorem 2. Assume that $u_0, u_1 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$, and the nonlinearity f has the form (16).

Then, there exists $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \|u_1\|_{\mathcal{M}^{p,1}_{0,s}})$ such that (2) has a unique solution $u \in C([0,T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$. Moreover, if $T^* < \infty$, then $\limsup_{t \to T^*} \|u(\cdot,t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$

Theorem 3. Assume that $u_0, u_1 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$, and the nonlinearity f has the form (16).

Then, there exists $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \|u_1\|_{\mathcal{M}^{p,1}_{0,s}})$ such that (3) has a unique solution u e $C([0,T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$. Moreover, if $T^* < \infty$, then $\limsup_{t \to T^*} \|u(\cdot,t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$

Remark 1. In Theorem 1 we can replace the (NLS) equation with the following more general (NLS) type equation:

$$(NLS)_\alpha \quad i\frac{\partial u}{\partial t} + \Delta_x^{\alpha/2} u + f(u) = 0, \; u(x,0) = u_0(x) \qquad (20)$$

for any $\alpha \in [0,2]$ and $p \geq 1$. The operator $\Delta_x^{\alpha/2}$ is interpreted as a Fourier multiplier operator (with t fixed), $\widehat{\Delta_x^{\alpha/2} u}(\xi,t) = |\xi|^\alpha \widehat{u}(\xi,t)$.

Remark 2. Theorems 1 and 2 of are particular cases of Theorem 1 with $p = 2$ and $s = 0$

COMPARISONS BETWEEN SINE-GORDON AND PERTURBED NONLINEAR SCHRÖDINGER EQUATIONS

The propagation and cooperation of spatially localized pulses (the alleged light bullets) with particle highlights in a few space measurements are of both physical and mathematical interests. Such light bullets have been seen in the numerical reenactments of the full Maxwell system with immediate Kerr ($\chi^{(3)}$ or cubic) nonlinearity in two space measurements (2D). They are short femtosecond pulses that spread without basically changing shapes over a long separation and have just a couple EM (electromagnetic) oscillations under their envelopes. They have been discovered valuable as data transporters in correspondence, as vitality sources, switches and rationale doors in optical gadgets.

In one space measurement (1D), the Maxwell system modeling light engendering in nonlinear media concedes steady speed voyaging waves as careful solutions, otherwise called the light bubbles (unipolar pulses or solitons). The complete integrability of a Maxwell-Bloch system. In a few space measurements, consistent velocity voyaging waves (mono-scale solutions) are harder to stop by. Rather, space-time swaying (different scale) solutions are more strong. The alleged light bullets are of numerous scale structures with particular stage/bunch speeds and adequacy elements. Despite the fact that direct numerical reproductions of the full Maxwell system are inspiring, asymptotic estimate is vital for analysis in a few space measurements. The estimate of 1D Maxwell system has been broadly examined. Long pulses are all around approximated through envelope guess by the cubic centering nonlinear Schrodinger (NLS) for $\chi^{(3)}$ medium. A correlation between Maxwell solutions and those of a broadened NLS likewise demonstrated that the cubic NLS estimate works sensibly well on short stable 1D pulses. Mathematical analysis on the legitimacy of NLS estimation of pulses and counter-spreading pulses of 1D sine-Gordon equation has been completed. Be that as it may, in 2D, the envelope estimate with the cubic centering NLS separates, in light of the fact that basic breakdown of the cubic centering NLS happens in limited time. Then again, because of the characteristic physical component or material reaction, Maxwell system itself regularly carries on fine past the cubic NLS breakdown time. One case is the semi-traditional two level dissipationless Maxwell-Bloch system where smooth solutions continue until the end of time. It is subsequently an exceptionally intriguing inquiry how to alter the cubic NLS guess to catch the right physics for modeling the engendering and collaboration of light flags in 2D Maxwell sort systems.

As of late, by looking at a recognized asymptotic point of confinement of the two level dissipationless Maxwell-Bloch system in the transverse electric administration, Xin (2000) found that the surely understood (2 + 1) sine-Gordon (SG) equation

$$\partial_{tt} u - c^2 \nabla^2 u + \sin(u) = 0, \quad t > 0, \tag{21}$$

with initial conditions

$$u(\mathbf{x}, 0) = u^{(0)}(\mathbf{x}), \quad \partial_t u(\mathbf{x}, 0) = u^{(1)}(\mathbf{x}), \quad \mathbf{x} = (x, y) \in \mathbb{R}^2 \tag{22}$$

where $u := u(\mathbf{x}, t)$ is a real-valued function and c is a given

constant, has its own light bullets solutions. It is well known that the energy

$$E^{SG}(t) := \int_{\mathbb{R}^2} \left[(\partial_t u)^2 + c^2 |\nabla u|^2 + 2G(u)\right] d\mathbf{x}, \quad t \geq 0 \tag{23}$$

with

$$G(u) = \int_0^u \sin(s)ds = 1 - \cos(u) \tag{24}$$

is preserved in the above SG equation. Direct numerical reproductions of the SG in 2D were performed, which are much less difficult undertakings than mimicking the full Maxwell. Moving heartbeat solutions having the capacity to keep the general profile over quite a while were watched, much the same as those in Maxwell system. See additionally for related breather-sort solutions of SG in 2D taking into account a modulation analysis in the Lagrangian plan. As per Xin (2000), another and complete perturbed NLS equation was determined by evacuating all reverberation terms (complete NLS guess) in doing the envelope development of SG. The new equation is second request in space-time and contains a nonparaxiality term, a blended subordinate term, and a novel nonlinear term which is immersing for expansive plentifulness. The equation is globally all around postured and does not have limited time collapse.

The principle motivation behind this study is to do extensive and precise numerical examinations between the arrangement of the SG equation and the solutions of the complete perturbed NLS and its limited term estimate in nonlinearity, and additionally the standard basic NLS. The calculation challenge required in SG recreation is that the dissimilar time scales amongst SG and perturbed NLS equations require a long-term reproduction of SG equation in an expansive 2D domain, which should be expanded if the intrigued time call attention to out to be further away because of the proliferating property of SG light shot solutions. Then again, for the perturbed NLS reenactment the test is that high spatial determination is required to catch the centering defocusing system which keeps the basic NLS breakdown. Here, semi-certain sine pseudospectral discretizations are proposed, which can be expressly tackled in stage and are of otherworldly exactness in space. Our outcomes give a numerical legitimization of the perturbed NLS as a substantial guess to SG in 2D, particularly past the breakdown time of cubic centering NLS.

Perturbed NLS and its approximations -

As derived in , we look for a modulated planar pulse solution of SG (1) in the form:

$$u(\mathbf{x}, t) = \varepsilon A(\varepsilon(x - vt), \varepsilon y, \varepsilon^2 t) e^{i(kx - \omega(k)t)} + \text{c.c.} + \varepsilon^3 u_2,$$
$$\mathbf{x} = (x, y) \in \mathbb{R}^2, \quad t \geq 0, \tag{25}$$

where $0 < \varepsilon \ll 1, \omega = \omega(k) = \sqrt{1 + c^2k^2}, v = \omega'(k) = c^2k/\omega$ the group velocity, and c.c. refers to the complex conjugate of the previous term. Plugging (25) into (21), setting $X = \varepsilon(x - vt)$, $Y = \varepsilon y$ and $T = \varepsilon^2 t$, calculating derivatives, expressing the sine function in series and removing all the resonance terms, one obtains the following complete perturbed NLS:

$$-2i\omega\partial_T A + \varepsilon^2 \partial_{TT} A = \frac{c^2}{\omega^2}\partial_{XX} A + c^2 \partial_{YY} A + 2\varepsilon v \partial_{XT} A$$
$$+ |A|^2 A \sum_{l=0}^{\infty} \frac{(-1)^l (\varepsilon |A|)^{2l}}{(l+1)!(l+2)!}, \quad T > 0, \tag{26}$$

where $A := A(\mathbf{X}, T), \mathbf{X} = (X, Y) \in \mathbb{R}^2$, is a complex-valued function.

Introducing the scaling variables $\tilde{X} = (\omega/c)X, \tilde{Y} = Y/c$ and $\tilde{T} = T/(2\omega)$, substituting them into (26) and then removing all $\tilde{\ }$, one gets a standard perturbed NLS,

$$i\partial_T A - \frac{\varepsilon^2}{4\omega^2}\partial_{TT} A = -\nabla^2 A - \frac{\varepsilon ck}{\omega}\partial_{XT} A + f_\varepsilon\left(|A|^2\right)A, \quad T > 0 \tag{27}$$

with initial conditions,

$$A(\mathbf{X}, 0) = A^{(0)}(\mathbf{X}), \quad \partial_T A(\mathbf{X}, 0) = A^{(1)}(\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^2 \tag{28}$$

where,

$$\rho = |A|^2, \quad f_\varepsilon(\rho) = \sum_{l=0}^{\infty} \frac{(-1)^{l+1}\varepsilon^{2l}\rho^{l+1}}{(l+1)!(l+2)!} \tag{29}$$

In fact, the Eq. (27) can be viewed as a perturbed cubic NLS with both a saturating nonlinearity (series) term and nonparaxial terms (the A_{TT} and A_{XT} terms). As proven by Xin (2000), it conserves the *energy*, i.e.,

$$E^{PNLS}(T) := \int_{\mathbb{R}^2} \left[\frac{\varepsilon^2}{4\omega^2} |A_T|^2 + |\nabla A|^2 + F_\varepsilon \left(|A|^2\right) \right] d\mathbf{X} \equiv E^{PNLS}(0), \quad T \geq 0, \tag{30}$$

with

$$F_\varepsilon(\rho) = \int_0^\rho f_\varepsilon(s) ds = \sum_{l=0}^{\infty} \frac{(-1)^{l+1} \varepsilon^{2l} \rho^{l+2}}{(l+1)!(l+2)!(l+2)} \tag{31}$$

and has the *mass* balance identity

$$\frac{d}{dT} \left(\int_{\mathbb{R}^2} |A|^2 d\mathbf{X} - \frac{\varepsilon^2}{2\omega^2} \operatorname{Im} \int_{\mathbb{R}^2} A_T \bar{A} d\mathbf{X} \right) = \frac{2\varepsilon \mathbf{v}}{c} \operatorname{Im} \int A_X \overline{A_T} d\mathbf{X}, \tag{32}$$

where $\bar{f}$ denotes the conjugate of *f*. In addition, the perturbed NLS (27) is globally well posed and does not have finite-time collapse , i.e., for any given initial data $A^{(0)}(\mathbf{X}) \in H^2(\mathbb{R}^2)$ and $A^{(1)}(\mathbf{X}) \in H^1(\mathbb{R}^2)$, the initial value problem of(27) with initial conditions (28) has a unique global solution $A \in C([0,\infty]; H^2(\mathbb{R}^2))$,

$A_T \in C([0,\infty]; H^1(\mathbb{R}^2))$, and $A_{TT} \in C([0,\infty]; L^2(\mathbb{R}^2))$.

In practice, the infinite series of the nonlinearity in (27) is usually truncated to finite terms with focusing-defocusing cycles. Denote

$$f_\varepsilon^N(\rho) = \sum_{l=0}^{N} \frac{\varepsilon^{4l} \rho^{2l+1}}{(2l+1)!(2l+2)!} \left[-1 + \frac{\varepsilon^2 \rho}{(2l+2)(2l+3)} \right] \tag{33}$$

then the perturbed NLS (27) can be approximated by the following truncated NLS:

$$i\partial_T A - \frac{\varepsilon^2}{4\omega^2} \partial_{TT} A = -\nabla^2 A - \frac{\varepsilon c k}{\omega} \partial_{XT} A + f_\varepsilon^N \left(|A|^2\right) A, \quad T > 0. \tag{34}$$

Similar as the proof in Xin (2000) for perturbed NLS (27), one can show that the truncated NLS (34) with the initial conditions (28) also conserves the *energy*, i.e.,

$$E_N^{PNLS}(T) := \int_{\mathbb{R}^2} \left[\frac{\varepsilon^2}{4\omega^2} |A_T|^2 + |\nabla A|^2 + F_\varepsilon^N \left(|A|^2\right) \right] d\mathbf{X} \equiv E_N^{PNLS}(0), \quad T \geq 0. \tag{35}$$

with

$$F_\varepsilon^N(\rho) = \int_0^\rho f_\varepsilon^N(s) ds = \sum_{l=0}^{N} \frac{\varepsilon^{4l} \rho^{2l-2}}{(2l+1)!(2l+2)!(2l+2)} \left[-1 + \frac{\varepsilon^2 \rho}{(2l+3)^2} \right] \tag{36}$$

and has the *mass* balance identity (32).

When $\varepsilon = 0$, the perturbed NLS (27) and its approximation (34) collapse to the well-known cubic (critical) focusing NLS:

$$i\partial_T A = -\nabla^2 A - \frac{1}{2}|A|^2 A, \quad T > 0 \tag{37}$$

with initial condition,

$$A(\mathbf{X}, 0) = A^{(0)}(\mathbf{X}), \quad \mathbf{X} \in \mathbb{R}^2 \tag{38}$$

It is well known that this cubic NLS conserves the energy, i.e.,

$$E^{CNLS}(T) := \int_{\mathbb{R}^2} \left[|\nabla A|^2 - \frac{1}{4}|A|^4 \right] d\mathbf{X} \equiv \int_{\mathbb{R}^2} \left[|\nabla A^{(0)}|^2 - \frac{1}{4}|A^{(0)}|^4 \right] d\mathbf{X}, \tag{39}$$

and when the initial energy $E^{CNLS}(0) < 0$, finite time collapse occurs in this focusing cubic (critical) NLS, which motivates different choices of initial data in (28) and (38) for our numerical experiments.

We remark here that as mentioned in the introduction, noting (25) the disparate time scales for the perturbed NLS equations (34) and the SG equation (21) are $T = O(1)$ and $t = O(\varepsilon^{-2})$, respectively, which immediately implies that it requires a much longer time simulation for the SG equation (21) if the time regime beyond the collapse time of the critical NLS (37) is of interest, when ε is small.

CONCLUSION

This thesis is devoted to numerical methods, their analysis and their applications, for some classes of nonlinear dispersive equations, namely the Schrodinger–Poisson–Slater (SPS) equation, the nonlinear relativistic Hartree equation, the nonlinear Klein–Gordon (KG) equation in the nonrelativistic limit regime, the sine–Gordon (SG) equation and perturbed NLS equation for modeling 2D light bullets.

REFERENCES

A.M. Wazwaz (2002). New solitary-wave special solutions with solitary patterns for the nonlinear dispersive (m, n) equations. Chaos, Solitons and Fractals, 13: pp. 161-170.

Benci, V., Fortunato, D. (1998). An eigenvalue problem for the Schr□bdinger- Maxwell equations. Topol. Methods Nonlinear Anal. 11: pp. 283-293.

C.Bardos, L. Erd¨os, F. Golse, N.J. Mauser, H.T. Yau (2002). Derivation of the Schr¨odinger-Poisson equation from the quantum N-particle Coulomb problem, C. R. Acad. Sci. Paris, Ser. I 334: pp. 515–520.

Chorin A.J. (1993). Numerical Study of Slightly Viscous Fiow, J. Fluid Mech., 57, pp. 785-796.

D'Aprile, T., Mugnai, D. (2004). Non-existence results for the coupled Klein-Gordon-Maxwell equations. Adv. Nonlinear Stud. 4: pp. 307-322.

D'Aprile, T., Mugnai, D. (2004). Non-existence results for the coupled Klein{Gordon{Maxwell equations. Adv. Nonlinear Stud. 4: pp. 307-322.

D'Aprile, T., Mugnai, D. (2004). Solitary Waves for nonlinear Klein- Gordon{Maxwell and Schrodinger{Maxwell equations. Proc. R. Soc. Edinb. Sect. A. 134: pp. 1-14.

E.Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type, to ap- pear in Mathematical Physics, Analysis and Geometry; arXiv:math.AP/0505456.

F.Linares, G. Ponce (2009). Introduction to Nonlinear Dispersive Equations, Springer, New York.

Holden, H., Lubich, C., and Risebro, N.H. (2013). Operator splitting for partial differential equations with Burgers nonlinearity, Math. Comp. 82, pp. 173–185.

Olver, P.J. (2010). Dispersive quantization, Amer. Math. Monthly 117, pp. 599–610.

T.Motai (2008). On the Cauchy problem for the nonlinear Klein Gordon equation with a cubic convolution, Tsukuba J. Math., 12, pp. 353-369.

Y.Zhu and X. Gao (2006). Exact special solitary solutions with compact support for the nonlinear dispersive (m, n) equations. Chaos, Solitons and Fractals, 27: pp. 487-493.

Z.M. Odibat (2007). Solitary solutions for the nonlinear dispersive (m, n) equations with fractional time derivatives. Physics Letters A, 370: pp. 295-301.

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