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Constant Coefficients Linear Higher Order Differential-Algebraic Equations

Shrikrishna Kakade[1]* Dr. Bhausaheb Sontakke[2]

[1] PhD Student, Kalinga University, Raipur

[2] PhD Guide, Kalinga University, Raipur

Abstract – This paper contributes to the theoretical analysis of linear Differential Algebraic Equations of higher order as well as of the regularity and singularity of matrix polynomials. Some invariants and condensed forms under appropriate equivalent transformations are given for systems of linear higher-order Differential-Algebraic Equations' with constant and variable coefficients. Inductively, based on condensed forms the original Differential-Algebraic Equations system can be transformed by differentiation-and-elimination steps into an equivalent strangeness-free system, from which the solution behaviour (including consistency of initial conditions and unique solvability) of the original Differential-Algebraic Equations system and related initial value problem can be directly read off. It is shown that the following equivalence holds for a Differential-Algebraic Equations system with strangeness-index ^ and square and constant coefficients. For any consistent initial condition and any right-hand side $f(t) \in C^{\mu}([t_0,t_1],\mathbb{C}^n)$ the associated initial value problem has a unique solution if and only if the matrix polynomial associated with the system is regular.

Some necessary and sufficient conditions for column- and row- regularity and singularity of rectangular matrix polynomials are derived. A geometrical characterization of singular matrix pencils is also given. Furthermore, an algorithm is presented which - using rank information about the coefficients matrices and via computing determinants - decides whether a given matrix polynomial is regular.

Keywords: Constant, Coefficients, Linear, Higher-Order, Differential-Algebraic, Equations, matrix, polynomial, etc.

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INTRODUCTION

In this chapter, we consider systems of linear /th-order $(l \geq 2)$ differential-algebraic equations with constant coefficients of the form

$$A_l x^{(l)}(t) + A_{l-1}x^{(l-1)}(t) + \cdots + A_0 x(t) = f(t),\ t \in [t_0, t_1], \quad (1)$$

where $A_i \in \mathbb{C}^{m\times n},\ i = 0,1,\ldots,l,\ A_l \neq 0,\ f(t) \in \mathcal{C}^{\mu}([t_0,t_1],\mathbb{C}^m)$, possibly together with initial conditions

$$x(t_0) = x_0,\ \ldots,\ x^{(l-2)}(t_0) = x_0^{[l-2]},\ x^{(l-1)}(t_0) = x_0^{[l-1]};\quad x_0,\ldots,x_0^{[l-2]},x_0^{[l-1]} \in \mathbb{C}^n. \quad (2)$$

Here, the nonnegative integer μ is the *strangeness-index* of the system (1), i.e., to get continuous solutions of the (1), the right-hand side $f(t)$ has to be μ-times continuously differentiable (later, in Section 2 we shall give an explicit definition of the strangeness-index).

First, let us clarify the concepts of *solution* of the system (1), *solution* of the initial value problem (1)-(2), and *consistency* of the initial conditions (2).

Definition 1: *A vector-valued function* $x(t) := [x_1(t),\ldots,x_n(t)]^T \in \mathcal{C}([t_0,t_1],\mathbb{C}^n)$ *is called* solution *of* (1) *if* $\sum_{k=1}^{n} A_i(j,k)x_k^{(i)}(t),\ i = 0,\ldots,l,\ j = 1,\ldots,m,$ *exist and for* j = 1,..., m *the following equations are satisfied:*

$$\sum_{k=1}^{n} A_l(j,k)x_k^{(l)}(t) + \sum_{k=1}^{n} A_{l-1}(j,k)x_k^{(l-1)}(t) + \cdots + \sum_{k=1}^{n} A_0(j,k)x_k(t) = f_j(t),$$

where A_j (j, k) *denotes the element of the matrix* A_j *lying on the* jth *row and the* kth *column of* Aj *and* f (t) := $[f_1(t),\ldots,f_m(t)]^T$.

A vector-valued function $x(t) \in \mathcal{C}(|t_0, t_1|, \mathbb{C}^n)$ *is called* solution of the initial value problem (1)-(2) *if it is a solution of* (1) *and, furthermore, satisfies* (2). *Initial conditions* (2.2) *are called* consistent *with the system* (1) *if the associated initial value problem* (1)-(2) *has at least one solution.*

In the last section we saw that DAEs differ in many ways from ordinary differential equations. For instance the circuit lead to a DAE where a differentiation process is involved when solving the equations. This differentiation needs to be carried out numerically, which is an unstable operation. Thus there are some problems to be expected when solving these systems. In this section we try to measure the difficulties arising in the theoretical and numerical treatment of a given DAE.

Modelling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, electrical circuits and chemical reaction kinetics.

In this paper we will give examples of how DAEs are obtained in these fields. We will point out important characteristics of differential-algebraic equations that distinguish them from ordinary differential equations.

Consider the (linear implicit) DAE system:

Ey' = A y + g(t) with consistent initial conditions and apply implicit Euler:

$$\mathbf{E(y_{n+1} - y_n)/h = A\, y_{n+1} + g(t_{n+1})}$$

and rearrangement gives:

$$\mathbf{y_{n+1} = (E - A\, h)^{-1} [E\, y_n + h\, g(t_{n+1})]}$$

Now the true solution, y(tn), satisfies:

$$\mathbf{E[(y(t_{n+1}) - y(t_n))/h + h\, y''(x)/2] = A\, y(t_{n+1}) + g(t_{n+1})}$$

and defining en = y(tn) - yn, we have:

$$\mathbf{e_{n+1} = (E - A\, h)^{-1} [E\, e_n - h_2\, y''(x)/2]}$$

$e_0 = 0$, known initial conditions where the columns of Aa correspond to the voltage, resistive and capacitive branches respectively. The rows represent the network's nodes, so that ¡1 and 1 indicate the nodes that are connected by each branch under consideration. Thus AA assigns a polarity to each branch.

Example 1 We investigate the initial value problem for the linear second-order constant coefficient Differential-Algebraic Equations'

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) = f(t), \quad t \in [t_0, t_1] \qquad (3)$$

Where $x(t) = [x_1(t), x_2(t)]^T$, and $f(t) = [f_1(t), f_2(t)]^T$ is suciently smooth, together with the initial conditions?

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_0^{[1]}, \qquad (4)$$

where $x_0 = [x_{01}, x_{02}]^T \in \mathbb{C}^2$, $x_0^{[1]} = [x_{01}^{[1]}, x_{02}^{[1]}]^T \in \mathbb{C}^2$. A short computation shows that system (2.3) has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t). \end{cases} \qquad (5)$$

Moreover, (5) is the unique solution of the initial value problem (3)-(4) if the initial conditions (4) are consistent, namely,

$$\begin{cases} x_{01} &= f_2(t_0), \\ x_{02} &= f_1(t_0) - \dot{f}_2(t_0) - \ddot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_2(t_0), \\ x_{02}^{[1]} &= \dot{f}_1(t_0) - \ddot{f}_2(t_0) - \left.\frac{d^3 f_2(t)}{dt^3}\right|_{t_0+}. \end{cases} \qquad (6)$$

If we let

$$v(t) = [v_1(t), v_2(t)]^T = [\dot{x}_1(t), \dot{x}_2(t)]^T, \; y(t) = [v_1(t), v_2(t), x_1(t), x_2(t)]^T,$$

then we have the following initial-value problem for the linear first-order Differential-Algebraic Equations'

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{y}(t) + \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} y(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ 0 \\ 0 \end{bmatrix}, \qquad (7)$$

together with the initial condition

$$y(t_0) = [x_{01}^{[1]}, x_{02}^{[1]}, x_{01}, x_{02}]^T. \qquad (8)$$

It is immediate that the system (7) of first-order Differential-Algebraic Equations has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t), \\ v_1(t) = \dot{f}_2(t), \\ v_2(t) = \dot{f}_1(t) - \ddot{f}_2(t) - f_2^{(3)}(t). \end{cases} \qquad (9)$$

In this form, (9) is the unique solution of the initial value problem (7)-(8) if the initial condition (8) is consistent, i.e.,

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$$\begin{cases} x_{01} &= f_2(t_0), \\ x_{02} &= f_1(t_0) - \dot{f}_2(t_0) - \ddot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_1(t_0) - \ddot{f}_2(t_0) - f_2^{(3)}(t_0). \end{cases} \quad (10)$$

Remark 4 Example 3 shows that the second-order system (3) has a unique continuous solution (5) if and only if the right-hand side satisfies

$$f(t) \in \mathcal{C}^2([t_0, t_1], \mathbb{C}^2),$$

whereas the converted first-order system (7) has a unique continuous solution if and only if $f(t) \in \mathcal{C}^3([t_0, t_1], \mathbb{C}^2)$; or in other words, the *strangeness-index* of the converted first-order system (7) is larger by one than that of the original second-order system (3). For a general system of l-th-order Differential-Algebraic Equations', it is not difficult to find similar examples.

Differential-Algebraic Equations' into an associated system of first-order Differential-Algebraic Equations' is not always equivalent in the sense that higher degree of the smoothness of the right-hand side f (t) may be involved in the solutions of the latter.

It should be noted that Example 3 also shows that, to obtain continuous solutions of a system of Differential-Algebraic Equations', some parts of the right-hand side f (t) may be required to be more differentiable than other parts which may be only required to be continuous; for a detailed investigation, we refer to, for example. Nonetheless, in order to simplify algebraic forms of a system of Differential-Algebraic Equations', we usually apply algebraic equivalence transformation to its matrix coefficients. For this reason and to avoid becoming too technical, we always consider the differentiability of the right-hand side vector-valued function f (t) as a whole, and do not distinguish the degrees of smoothness required of its components from each other.

MATRIX CONDENSED FORM:

As we have mentioned for convenience of notation and expression, in this section we shall work mainly with systems of linear second-order Differential-Algebraic Equations' with constant coefficients

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = f(t), \ t \in [t_0, t_1], \quad (11)$$

with $M, C, K \in \mathbb{C}^{m \times n}, \ f(t) \in \mathcal{C}^{\mu}([t_0, t_1], \mathbb{C}^m),$
possibly together with initial conditions

$$x(t_0) = x_0, \ \dot{x}(t_0) = x_0^{[1]}, \quad x_0, x_0^{[1]} \in \mathbb{C}^n. \quad (12)$$

It is well-known that the nature of the solutions of the system of linear first-order constant coefficient Differential-Algebraic Equations'

$$E\dot{x}(t) = Ax(t) + f(t), \ t \in [t_0, t_1],$$

with $E, A \in \mathbb{C}^{m \times n}$ and $f(t) \in \mathcal{C}^{\mu}([t_0, t_1], \mathbb{C}^m)$, can be determined by the properties of the corresponding matrix pencil $\lambda E - A$. Furthermore, the algebraic properties of the matrix pencil $\lambda E - A$ can be well understood through studying the canonical forms for the set of matrix pencils

$$\lambda(PEQ) - (PAQ), \quad (13)$$

where $P \in \mathbb{C}^{m \times m}, Q \in \mathbb{C}^{n \times n}$ are any nonsingular matrices. In particular, among those canonical forms for (13) are the well-known *Weierstrass canonical form* for regular matrix pencils and the *Kronecker canonical form* for general singular matrix pencils from which one can directly read off the solution properties of the corresponding Differential-Algebraic Equations'.

Similarly, as we will see later in this chapter, the behaviour of solutions of the system (11), as well as the initial value problem (11)-(12), depends on the properties of the quadratic matrix polynomial

$$A(\lambda) = \lambda^2 M + \lambda C + K. \quad (14)$$

If we let $x(t) = Qy(t)$, and premultiply (11) by P, where $P \in \mathbb{C}^{m \times m}, Q \in \mathbb{C}^{n \times n}$ are

nonsingular matrices, we obtain an *equivalent* system of Differential-Algebraic Equations'

$$(PMQ)\ddot{y}(t) + (PCQ)\dot{y}(t) + (PKQ)y(t) = Pf(t), \quad (15)$$

and a new corresponding quadratic matrix polynomial

$$\hat{A}(\lambda) = \lambda^2 \hat{M} + \lambda \hat{C} + \hat{K} := \lambda^2(PMQ) + \lambda(PCQ) + (PKQ).$$
(16)

Here, by *equivalence* we mean not only that the relation $x(t) = Qy(t)$ (or $y(t) = Q^{-1}x(t)$)

However, it is also well-known that it is an open problem to find a canonical form for quadratic matrix polynomials (16), let alone higher-degree matrix polynomials, from which we can *directly* read off the solution properties of the corresponding system of Differential-Algebraic Equations'. Nonetheless, inspired by the work of (though the papers mainly deal with linear first-order Differential-Algebraic Equations with *variable coefficients*), we shall in this section give an equivalent condensed form for quadratic matrix polynomials (14) through purely algebraic manipulations and coordinate changes. Based on the condensed form we can *partially* decouple the system into three parts, namely, an ordinary-differential-equation part, an algebraic part and a coupling part, and therefore pave the way for the further treatment of the system in the following section. Sometimes, we will use the notation $(A_l, \ldots, A_1, A_0)$ of a matrix (*l* + 1)-tuple instead of the matrix polynomial $\lambda^l A_l + \cdots + \lambda A_1 + A_0$ of lth degree which is associated with the general *l*th-order system (1) of Differential-Algebraic Equations'. By the following definition, we make the concept of *equivalence* between two general matrix (*l* + 1)-tuples clear.

Definition *Two* $(l+1)$-*tuples* $(A_l, \ldots, A_1, A_0)$ *and* $(B_l, \ldots, B_1, B_0)$, $A_i, B_i \in \mathbb{C}^{m\times n}$, $i = 0, 1, \ldots, l$, $l \in \mathbb{N}_0$, *of matrices are called* (strongly) equivalent *if there are non- singular matrices* $P \in \mathbb{C}^{m\times m}$ *and* $Q \in \mathbb{C}^{n\times n}$ *such that*

$$B_i = PA_iQ, \quad i = 0, 1, \ldots, l. \tag{17}$$

If this is the case, we write $(A_l, \ldots, A_1, A_0) \sim (B_l, \ldots, B_1, B_0)$.

The result on the canonical form for a single matrix under equivalence relation (17) is well-known:

Lemma: *let* $A \in \mathbb{C}^{m\times n}$, *then there are nonsingular matrices*

$$P : \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{C}^{m\times m} \text{ and } Q : \quad [Q_1, Q_2] \in \mathbb{C}^{n\times n}$$

Such that

$$PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \tag{18}$$

Where $P_1 \in \mathbb{C}^{r\times m}$, $Q_1 \in \mathbb{C}^{n\times r}$, *moreover, we have*

$$r = \text{rank}(A),\ \mathcal{N}(A) = \mathcal{R}(Q_2),\ \mathcal{N}(A^T) = \mathcal{R}(P_2^T), \tag{19}$$

Where N(■) denotes the null space of a matrix, and R(-) the column space of a matrix.

The condensed form for a matrix pair *(E,A)* under equivalence relation (17) has been implicitly.

Lemma *Let* $E, A \in \mathbb{C}^{m\times n}$, *and let*

(a) $Z_1 \in \mathbb{C}^{m\times(m-r)}$ *be a matrix whose columns form a basis for* $\mathcal{N}(E^T)$
(b) $Z_2 \in \mathbb{C}^{n\times(n-r)}$ *be a matrix whose columns form a basis for* $\mathcal{N}(E)$. (20)

Then, the matrix pair (E, *A) is equivalent to a matrix pair of the form*

$$\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) \begin{matrix} s \\ d \\ a \\ s \\ v \end{matrix}, \tag{21}$$

where $s, d, a, v \in \mathbb{N}_0$, $A_{14} \in \mathbb{C}^{s\times u}$, $u \in \mathbb{N}_0$, *and the quantities (in the following we use the convention* $\text{rank}(0) = 0$*)*

$$\begin{aligned} &(a) \quad r = \text{rank}(E) \\ &(b) \quad a = \text{rank}(Z_1^T A Z_2) \\ &(c) \quad s = \text{rank}(Z_1^T A) - a \\ &(d) \quad d = r - s \\ &(e) \quad v = m - r - a - s \\ &(f) \quad u = n - r - a \end{aligned} \tag{22}$$

are invariant under equivalence relation (17).

For completeness, we give a proof of this lemma.

Proof of Lemma. In the following, the word "new" on top of the equivalence operator denotes that the subscripts of the entries are adapted to the new block structure of the matrices. Using Lemma, we obtain the following sequence of equivalent matrix pairs.

$$(E, A) \sim \left(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right) \overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix} \right) \sim \left(\begin{bmatrix} I_r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & I_a & 0 \\ A_{31} & 0 & 0 \end{bmatrix} \right)$$

$$\overset{\text{new}}{\sim} \left(\begin{bmatrix} P_{11} & P_{12} & 0 & 0 \\ P_{21} & P_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} & 0 & A_{14} \\ A_{21} & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$\left(\text{where the matrix } \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \text{ is nonsingular} \right)$$

$$\overset{\text{new}}{\sim}\left(\begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right).$$

It remains to show that such quantities r, s, *d, a,* v, *u* are well-defined by (22) and invariant under the equivalence relation (17). In the case of r = rank (E), this is clear. For the other quantities, indeed, we only need to show two quantities *a* and *s* are well-defined and invariant under equivalence relation (17). Since we have proved

(2.21), let $P := \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \in \mathbb{C}^{m\times m}$ and $Q := [Q_1, Q_2] \in \mathbb{C}^{n\times n}$ be nonsingular matrices, where $P_1 \in \mathbb{C}^{r\times m}$, $Q_1 \in \mathbb{C}^{n\times r}$, such that

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} E[Q_1, Q_2] = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & I_d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} A[Q_1, Q_2] = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & I_a & 0 \\ I_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (23)$$

By Lemma, we have

$$\mathcal{N}(E^T) = \mathcal{R}(P_2^T), \quad \mathcal{N}(E) = \mathcal{R}(Q_2), \qquad (24)$$

namely, the columns of P_2^T span $\mathcal{N}(E^T)$, and the columns of Q_2 span $\mathcal{N}(E)$. From (23) it immediately follows that

$$P_2AQ = \begin{bmatrix} 0 & 0 & I_a & 0 & 0 \\ I_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad P_2AQ_2 = \begin{bmatrix} I_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (25)$$

Hence, by (25), we have

$$a = \operatorname{rank}(P_2AQ_2), \quad s = \operatorname{rank}(P_2AQ) - a = \operatorname{rank}(P_2A) - a. \qquad (26)$$

From (20) and (24) it follows that there exist nonsingular matrices $T_1 \in \mathbb{C}^{(m-r)\times(m-r)}$ and $T_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ such that

$$P_2^T = Z_1T_1, \quad Q_2 = Z_2T_2. \qquad (27)$$

Then, from (2.26) and (2.27) it follows that

$$\begin{aligned} a &= \operatorname{rank}(P_2AQ_2) = \operatorname{rank}\left(T_1^T Z_1^T A Z_2 T_2\right) = \operatorname{rank}(Z_1^T A Z_2), \quad \text{and} \\ s &= \operatorname{rank}(P_2A) - a = \operatorname{rank}\left(T_1^T Z_1^T A\right) - a = \operatorname{rank}(Z_1^T A) - a. \end{aligned}$$

Thus, *a* and s are indeed well-defined by (22) and therefore so are the quantities d, v and *u.*

Thus, we have prepared the way for further analyzing the systems (11) and (1) of Differential-Algebraic Equations', which will be presented in the next two sections.

LINEAR 1ST AND 2ND ORDER DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH CONSTANT COEFFICIENTS:

In this section, we discuss the system (11) of Differential-Algebraic Equations', and answer those questions raised at the beginning of this chapter. Let us start by writing down the system of differential-algebraic equations in the last section we saw that DAEs differ in many ways from ordinary differential equations. For instance the circuit lead to a DAE where a differentiation process is involved when solving the equations. This differentiation needs to be carried out numerically, which is an unstable operation. Thus there are some problems to be expected when solving these systems. In this section we try to measure the difficulties arising in the theoretical and numerical treatment of a given DAE.

Modelling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, electrical circuits and chemical reaction kinetics. In this section we will give examples of how DAEs are obtained in these fields.

We will point out important characteristics of differential-algebraic equations that distinguish them from ordinary differential equations. More information about differential-algebraic equations can be found but also in Consider the mathematical pendulum. By construction the rows of AA are linearly dependent. However, after deleting one row the remaining rows describe a set of linearly independent equations; the node corresponding to the deleted row will be denoted as the ground node.

As seen in the previous sections a DAE can be assigned an index in several ways. In the case of linear equations with constant coefficients all index notions coincide with the Kronecker index. Apart from that, each index definition stresses different aspects of the DAE under consideration. While the differentiation index aims at finding possible reformulations in terms of ordinary differential equations, the tractability index is used to study DAEs without the use of derivative arrays. In this section we made use of the sequence (2) established in the context of the tractability index in order to perform a refined analysis of linear DAEs with properly stated leading terms. We were able to find explicit expressions of (12) for these equations with index 1 and 2. Let me be the pendulum's mass which is attached to a rod of length l. In order to describe the pendulum in Cartesian coordinates we write down the potential energy U(x; y) = mgh = mgl ¡ mgy where ¡ x(t); y(t) ¢ is the position of the moving mass at time t. The earth's acceleration of gravity is given by g, the pendulum's height is h. If we denote derivatives of x and y by x˙ and y˙ respectively, the kinetic energy some additional simple examples:

Consider the (linear implicit) DAE system:

Ey' = A y + g(t) with consistent initial conditions and apply implicit Euler:

$$\mathbf{E(y_{n+1} - y_n)/h = A\ y_{n+1} + g(t_{n+1})}$$

and rearrangement gives:

$$\mathbf{y_{n+1} = (E - A\ h)^{-1}\ [E\ y_n + h\ g(t_{n+1})]}$$

Now the true solution, y(tn), satisfies:

$$\mathbf{E[(y(t_{n+1}) - y(t_n))/h + h\ y''(x)/2] = A\ y(t_{n+1}) + g(t_{n+1})}$$

and defining $e_n = y(t_n) - y_n$, we have:

$$\mathbf{e_{n+1} = (E - A\ h)^{-1}\ [E\ e_n - h_2\ y''(x)/2]}$$

$e_0 = 0$, known initial conditions where the columns of AA correspond to the voltage, resistive and capacitive branches respectively. The rows represent the network's nodes, so that ¡1 and 1 indicate the nodes that are connected by each branch under consideration. Thus AA assigns a polarity to each branch.

This detailed analysis leads us to results about existence and uniqueness of solutions for DAEs with low index. We were able to figure out precisely what initial conditions are to be posed, namely $D(t_0)x(t_0) = D(t_0)x_0$ and $D(t_0)P_1(t_0)x(t_0) = D(t_0)P_1(t_0)x_0$ in the index 1 and index 2 case respectively.

CONCLUSION:

In this paper we have presented the theoretical analysis of two interrelated topics: linear differential-algebraic equations of higher-order and the regularity and singularity of matrix polynomials.

In the first part of this paper, we have directly investigated the mathematical structures of general (including over- and underdetermined) linear higher-order systems of Differential-Algebraic Equations' with constant and variable coefficients. Making use of the algebraic techniques devised and taking linear second-order systems of Differential-Algebraic Equations' as examples, we have given condensed forms, under strong equivalence transformations, for triples of matrices and triples of matrix-valued functions which are associated with the systems of constant and variable coefficients respectively. It should be noted that in the case of variable coefficients, we have developed a system of invariant quantities and a set of regularity conditions to ensure that the condensed form can be obtained. Based on the condensed forms, we have converted the systems into ordinary-differential-equation part, 'strange' coupled differential-algebraic-equation part, and algebraic-equation part, and designed the differentiation-and-elimination steps to partially decouple the strange part. Inductively conducting such process of transformation and decoupling, we have, finally, converted the original systems into equivalent strangeness-free systems, from which the solution behaviour with respect to solvability, uniqueness of solutions and consistency of initial conditions can be directly read off. In the future we expect that detecting the regularity and singularity and providing information on the nearness to singularity will be realized in those software packages which deal with systems of linear differential-algebraic equations with constant coefficients and polynomial eigenvalue problems.

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Corresponding Author

Shrikrishna Kakade*

PhD Student, Kalinga University, Raipur

kakdeshrikrishna83@gmail.com