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G-Metric Space by using Clr G Property and Hybrid Contractive Conditions in Fixed Point Theory

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Abstract - Many writers have attempted to broaden the concept of a metric space, motivated by the fact that metric fixed point theory has applications in practically all branches of quantitative sciences. In this regard, numerous writers have proposed many generalised metric spaces in the previous decade. The concept of G-Metric space has piqued the interest of fixed point theorists among all the generalised metric spaces. Mustafa and Sims presented the notion of a G-Metric space in, where they examined the topological features of this space and established the analogue of the Banach contraction principle in G-metric spaces. Many writers have explored and proposed various common fixed point theorems in this framework as a consequence of these findings.

Keywords - G-Metric Space, CLR g Property, Hybrid Contractive Conditions, Fixed Point Theory, common fixed point theorems.

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INTRODUCTION

M.Aamri and D.El Moutawakil published the property (E.A) in 2002, which is a genuine generalisation of noncompatible mappings in metric spaces. Many popular fixed point theorems and their references were examined in the literature using this assumption. Sintunavarat et al. established the idea of the Common limit in the range of g (CLRg) property for a pair of self-mappings in Fuzzy metric space in 2011. The significance of this characteristic is that it assures that the proximity of range subspaces is not required, and as a consequence, writers are increasingly focusing on it in order to generalise conclusions found in the literature (see and the references therein). E.Karampur et al. have extended this to two pairs of self-mappings as the CLR(S,T) characteristic. We derive some common fixed point theorems in the realm of G-metric space, which generalizes various comparable results in and others, by employing the notions of common limit in the range property for two as well as four self-maps and weak compatibility, which is an efficient tool in providing the common fixed points. Simultaneously, we provide appropriate examples to demonstrate the applicability of the key findings. The fundamental definitions required in the major findings are listed below.

Definition 1 Let G: X X X [0,] be a function meeting the following conditions and X be a nonempty set:

G1 G(x, y, z) = 0 if x = y = z.

For every x, y X, G2 0 G(x, x, y) with x 6= y.

G3 G(x, x, y) G(x, y, z) G(x, y, z) G(x, y, z) for any x, y, z X with z 6= y.

G4 G(x, y, z) = G(x, z, y) = G(y, z, x) = G(y, z, x) = G(y, z, x) (symmetry in all three variables).

For any x, y, z, a, G5 G(x, y, z) G(x, a, a) + G(a, y, z) (rectangle inequality)

The function G is therefore known as a generalised metric, or a Gmetric on X, and the pair (X, G) is known as a G-metric space..

Definition 2 A G-Metric space (X, G) is said to be symmetric if G(x, y, y) = G(y, x, x) for all x, y ∈ X

Example 1: Consider the ordinary metric space (X, d). G(x, y, z) = maxd(x, y), d(y, z), d(z, x) for all x, y, z X is thus a G-Metric space.

Mustafa and Sims also verified the following statement in their first article.

Proposition 1. Let (X, G) be a G-metric space. Then, for any x, y, z, a ∈ X, it follows that

1. if G(x, y, z) = 0, then x = y = z

2. G(x, y, z) ≤ G(x, x, y) + G(x, x, z).

3. G(x, y, y) ≤ 2G(y, x, x)

4. G(x, y, z) ≤ G(x, a, z) + G(a, y, z).

5. G(x, y, z) ≤ 2 3 [G(x, y, a) + G(x, a, z) + G(a, y, z)].

6. G(x, y, z) ≤ G(x, a, a) + G(y, a, a) + G(z, a, a)

Definition 3- Let f, g be two metric space self-mappings (X, d). Then, if there exists a sequence xn in X such that limn fxn = limn gxn = t for any t X, we say that f and g fulfil the condition (E.A).

Definition 4 Let f, g, S, and T be four symmetric space self-mappings (X, d). The pairings (f, S) and (g, T) are said to have the common limit range property (with regard to S and T), frequently indicated by CLR(S,T), if there are two sequences xn and yn in X such that limn fxn = limn Sxn = limn gyn = limn gyn = limn gyn = limn gyn = limn g For some t, u, and w X, T yn = t with t = Su = Tw. Due to Sintunavarat et al., if f = g and S = T, the following formulation implies the CLRg property.

Main Result

The first result is an extended stringent contractive condition common fixed point theorem for a pair of self-mappings, which extends Theorem 1.

Theorem 1 Let's say (X,G) is a symmetric pair. f,g be two weakly compatible self-mappings on X fulfilling the G-Metric space.

1. CLRg property.

2. G(fx, fy, fz) < max{G(gx, gy, gz) G(fx,gx,gx)+G(fy,gy,gy)+G(fz,gz,gz) , G(fy,gx,gx)+G(fz,gy,gy)+G(fx,gz,gz) 3 } ∀x, y, z ∈ X with x 6= y.

Then f and g have a unique common fixed point.

Proof: 1 There exists a sequence xn in X such that f and g satisfy the CLRg condition. limn fxn = limn gxn = gx for some x ∈ X. Consider G(fxn, fx, fx) < max{G(gxn, gx, gx), G(fxn,gxn,gxn)+G(fx,gx,gx)+G(fx,gx,gx) 3 , $\frac{G(fz,gz,gz)+G(fx,gx,gx)+G(fx,gx,gx)}{3}$,

Letting n → ∞, we obtain G(gx, fx, fx) $\leq \frac{2}{3}G(gx, fx, fx)$ which implies gx = fx. Thus x is the coincidence point of f and g. Let z = fx = gx.

Since (f, g) are weakly compatible, we have fz = fgz = gfz = gz.

Now we will prove that fz = z. Suppose fz 6= z, then

G(fz, z, z) = G(fz, fx, fx) < max{G(gz, gx, gx) $\frac{G(fz,gz,gz)+G(fx,gx,gx)+G(fx,gx,gx)}{3}$, $\frac{G(fz,gz,gz)+G(fx,gx,gx)+G(fx,gx,gx)}{3}$, < G(fz, z, z), which is a contradiction.

Hence fz = z = gz. Thus z is the common fixed point of f and g. The uniqueness of the fixed point can be proved easily.

We now illustrate this theorem by giving an example.

Example 2: Let X = and G : X × X × X → [0,∞) defined by G(x, y, z) = 0 if x = y = z and G(x, y, z) = max{x, y, z} in all other cases. Then (X, G) is a symmetric G-Metric space. Let f, g be two self-maps on X defined by

fx = 5 if x ≤ 5 , fx = 3 if x > 5 and gx = $\frac{x+5}{2}$ if x ≤ 5 , gx = 10 if x > 5. Here f and g satisfies the CLRg property. To see this, consider a sequence

$\{x_n\} = \{5 - \frac{1}{n}\}\ \forall n.$ Then $fx_n = f(5 - \frac{1}{n}) \to 5$ and $gx_n = g(5 - \frac{1}{n}) = \frac{5 - \frac{1}{n} + 5}{2} \to$

5. Therefore limn fxn = limn gxn = 5 = g5

Further, (f, g) are weakly compatible and

G(fx, fy, fz) < max{G(gx, gy, gz), $\frac{G(fx,gx,gx)+G(fy,gy,gy)+G(fz,gz,gz)}{3}$, $\frac{G(fy,gx,gx)+G(fz,gy,gy)+G(fx,gz,gz)}{3}$} ∀x, y, z ∈ X with x 6= y. Thus f and g satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at x = 5.

In 1977, Mathkowski introduced the Φ-map as the following: Let Φ be the set of auxiliary functions φ such that φ : [0,∞) → [0, ∞) is a nondecreasing function satisfying limn φ n (t) = 0 for all t ∈ (0,∞). If φ ∈ Φ, then φ is called a Φ-map. Further φ(t) < t for all t ∈ (0,∞) and φ(0) = 0. We found a unique shared fixed point for two pairs of self-mappings involving a -map under the lipschitz kind of contractive condition in the following result. This finding broadens and generalises M.Aamri and El Moutawakil's Theorem 2 and S Manro et al's Theorem 1.

Theorem .1 Let (X,G) let a symmetric G-Metric space, and f,g,S,T be four self-mappings on X, as follows:

1. (f,S) and (g,T) satisfies CLR(S,T) property.

2. G(fx, gy, gz) ≤ φ(max{G(Sx, T y, T z), G(Sx, gy, gz), G(T y, gy, gz), G(gy, T y, T z)}) ∀x, y, z ∈ X

3. (f,S) and (g,T) are weakly compatible.

Then f ,g,S and T have a unique common fixed point.

Proof: Since (f, S) and (g, T) satisfies CLR(S,T) property, there exists two sequences {xn} and {yn} such that limn fxn = limn Sxn = limn gyn = limn T yn = t with t = Sx = T y for some t, x, y ∈ X. Consider, G(fx, gyn, gyn) ≤ φ(max{G(Sx, T yn, T yn), G(Sx, gyn, gyn), G(T yn, gyn, gyn), G(gyn, T yn, T yn)}) on letting n → ∞, we obtain G(fx, t, t) ≤ φ(0) = 0 which implies fx = t = Sx. Hence x is the coincidence point of f and S. Since (f, S) are weakly compatible, we have ffx = fSx = Sfx = SSx.

Now we prove that T y = gy. Consider, G(fxn, gy, gy) ≤ φ(max{G(Sxn, T y, T y), G(Sxn, gy, gy), G(T y, gy, gy), G(gy, T y, T y)})

As n → ∞, we have G(t, gy, gy) ≤ φ(G(t, gy, gy)) which implies G(t, gy, gy) = 0. Therefore gy = t = T y. i.e. y is the coincidence point of g and T.

Since (g, T) are weakly compatible we have ggy = gT y = T gy = T T y. Also note that fx = Sx = gy = T y = t.

Now we prove that ffx = fx. Suppose fx 6= ffx, then

G(ffx, fx, fx) = G(ffx, gy, gy) ≤ φ(max{G(Sfx, T y, T y), G(Sfx, gy, gy), G(T y, gy, gy), G(gy, T y, T y)}) < G(ffx, fx, fx), a contradiction.

Hence ffx = fx = Sfx, which implies fx is the common fixed point of f and S. Similarly one can prove gy is the common fixed point of g and T. Since fx = gy, z = fx is the common fixed point of f, g, S and T. The uniqueness of the fixed point follows easily.

As a corollary of Theorem 3, we derive the following sharpened version of Theorem contained in S.Manro, as conditions on the ranges of involved mappings are relatively lightened.

Corollary 1 Let (X,G) be a symmetric G-Metric space and f,g,S,T be four self-mappings on X such that

1. (f,S) and (g,T) satisfies CLR(S,T) property.

2. G(fx, gy, gy) ≤ φ(max{G(Sx, T y, T y), G(Sx, gy, gy), G(T y, gy, gy), G(gy, T y, T y)}) ∀x, y ∈ X.

3. (f,S) and (g,T) are weakly compatible.

Then f ,g,S and T have a unique common fixed point.

Proof: Put z = y in Theorem 1

By restricting f, g, S, T suitably, one can derive the corollaries involving two as well as three self-mappings as follows:

Corollary 2 Let (X,G) be a symmetric G-Metric space and f,g,S be three self-mappings on X such that

1. (f,S) and (g,S) satisfies CLRS property.

2. G(fx, gy, gz) ≤ φ(max{G(Sx, Sy, Sz), G(Sx, gy, gz), G(Sy, gy, gz), G(gy, Sy, Sz)}) ∀x, y, z ∈ X.

3. (f,S) and (g,S) are weakly compatible.

Then f ,g and S have a unique common fixed point.

Proof:2 Follows from Theorem 4 by setting S = T

RESULTS ON G-METRIC SPACE BY USING CLR_G PROPERTY

The first result extends Theorem 1 of with a generalised stringent contractive condition.

Theorem 2. Let f and g be self-mappings of a symmetric G-metric space (Y, G) that are weakly compatible. CLRg property and

$$G(fy_1, fy_2, fy_3) < \max\Big\{G(gy_1, gy_2, gy_3), \frac{G(fy_1, gy_1, gy_1) + G(fy_2, gy_2, gy_2) + G(fy_3, gy_3, gy_3)}{3}, \frac{G(fy_2, gy_1, gy_1) + G(fy_3, gy_2, gy_2) + G(fy_1, gy_3, gy_3)}{3}\Big\},$$ ………

…..(1)

∀ y1, y2, y3 ∈ Y . Then f and g have a unique common fixed point.

Proof. 3 The CLRg property is defined as a sequence n in Y such that limn→∞ fαn = limn→∞ gαn = gα for some α ∈ Y . Consider

$$G(f\alpha_n, f\alpha, f\alpha) < \max\Big\{G(g\alpha_n, g\alpha, g\alpha), \frac{G(f\alpha_n, g\alpha_n, g\alpha_n) + G(f\alpha, g\alpha, g\alpha) + G(f\alpha, g\alpha, g\alpha)}{3}, \frac{G(f\alpha, g\alpha_n, g\alpha_n) + G(f\alpha, g\alpha, g\alpha) + G(f\alpha_n, g\alpha, g\alpha)}{3}\Big\}.$$

…......................(2)

On letting n → ∞, we obtain G(gα, fα, fα) ≤ 2 3G(gα, fα, fα) which implies gα = fα. As a result, f and g coincide at this location. Let r = fα = gα.

We have (f, gpoor)'s compatibility. fr = fgα = gfα = gr.

To prove fr = r: Suppose fr 6= r, then

$$G(fr, r, r) = G(fr, f\alpha, f\alpha) < \max\Big\{G(gr, g\alpha, g\alpha), \frac{G(fr, gr, gr) + G(f\alpha, g\alpha, g\alpha) + G(f\alpha, g\alpha, g\alpha)}{3}, \frac{G(f\alpha, gr, gr) + G(f\alpha, g\alpha, g\alpha) + G(fr, g\alpha, g\alpha)}{3}\Big\}$$

< G(fr, r, r), a contradiction.

The shared fixed point of f and g is hence r. The fixed point's uniqueness is simply shown.

We will now provide an example to demonstrate this theorem..

Example 3. Let Y = [2, 20] and G : Y × Y × Y → [0,∞) defined by

$$G(y_1, y_2, y_3) = \begin{cases} 0 & \text{if} \quad y_1 = y_2 = y_3 \\ \max\{y_1, y_2, y_3\} & \text{in all other cases .} \end{cases}$$

Then (Y, G) is a symmetric G-metric space. Define f, g : Y → Y by

$$fy = \begin{cases} 5 & \text{if} \quad y \leq 5 \\ 3 & \text{if} \quad y > 5 \end{cases} \qquad \text{and} \qquad gy = \begin{cases} \frac{y+5}{2} & \text{if} \quad y \leq 5 \\ 10 & \text{if} \quad y > 5\,. \end{cases}$$

Here f and g satisfies the CLRg property. To see this, consider a sequence $\{\alpha_n\} = \{5 - \frac{1}{n}\}$ for all n

Then fαn = $f\left(5 - \frac{1}{n}\right) \to 5$ and $g\alpha_n = g\left(5 - \frac{1}{n}\right) = \frac{5 - \frac{1}{n} + 5}{2} \to 5.$

Therefore $\lim_{n\to\infty} f\alpha_n = \lim_{n\to\infty} g\alpha_n = 5 = g5.$

Further, (f, g) are weakly compatible satisfying (1) and y = 5 is the unique common fixed point.

We prove a fixed point theorem for two pairs of self-mappings involving a -map under the Lipschitz type of contractive condition in the following result, which extends and generalizes the previous conclusion. Theorems 1 and 2 are two of the most important theorems in mathematics.

INTEGRAL TYPE OF CONTRACTIVE CONDITION UNDER F-WEAK RECIPROCAL CONTINUITY

Branciari established the integral form of contractive condition in fixed point theorems. Aydi further expanded these findings to the class of G-metric spaces, while Shatanawi created G-metric space maps. Shatanawi et al. used the notion of maps with integral type contractions to get some intriguing findings in G-metric space in their paper.

Let Ψ be the set of functions ϕ : [0,∞) → [0, ∞), where ϕ is a lebesgue integrable mapping, summable, non-negative and ∀ , a, b > 0,

$$\int_0^{\epsilon} \varphi(y)dy > 0 \qquad \text{and} \qquad \int_0^{a+b} \varphi(y)dy \leq \int_0^{a} \varphi(y)dy + \int_0^{b} \varphi(y)dy.$$

We begin by defining f-weakly reciprocally continuous maps, which is a more extended version of f-reciprocal continuity..

Definition 5 . A G-metric space (Y, G) has two self-mappings f and g that are f-weakly reciprocal.

continuous iff $\lim_{n\to\infty} fg\alpha_n = fs$ or $\lim_{n\to\infty} gg\alpha_n = gs$

whenever {αn} is a sequence in Y such that $\lim_{n\to\infty} f\alpha_n = \lim_{n\to\infty} g\alpha_n = s$ for some s in Y .

Example Let Y = and G : Y × Y × Y → [0,∞) defined by G(y1, y2, y3) = max{d(y1, y2), d(y2, y3), d(y3, y1)}, where d is the usual metric on Y . Then (Y, G) is a G-metric space. Define f, g : Y → Y by

$$fy = \frac{y+3}{2} \text{ if } y < 6, \quad fy = 3 \text{ if } y \geq 6,$$

g3 = 3, gy = 10 if y < 3 and 3 < y < 6, $gy = \frac{y}{2} \text{ if } y \geq 6.$

Let $\{\alpha_n\} = \{6 + \frac{1}{n}\}$ be a sequence in Y. Then $f\alpha_n \to 3$, $g\alpha_n = 3 + \frac{1}{2n} \to 3$,

$$fg\alpha_n = f(3 + \tfrac{1}{2n}) \to 3 \text{ and } gg\alpha_n = g(3 + \tfrac{1}{2n}) \to 10.$$

Thus limn→∞ fgαn = f3 but $\lim_{n\to\infty} gg\alpha_n \neq g3.$ As a result, f and g are reciprocally weakly continuous but not reciprocally continuous..

Remark . As seen in the case above, f-weak reciprocal continuity implies f-reciprocal continuity, but not the other way around..

Theorem 7. Let f and g be f-weakly reciprocally continuous self-maps of a complete G-metric space Y with fY ⊆ gY . Let φ ∈ Φ satisfying

$$\int_0^{G(fy_1, fy_2, fy_3)} \varphi(y)dy \leq \phi\left(\int_0^{L(y_1, y_2, y_3)} \varphi(y)dy\right), \qquad \ldots$$

........(1)

∀ y1, y2, y3 ∈ Y , where ϕ ∈ Ψ and

$$L(y_1, y_2, y_3) = \max\{G(gy_1, fy_2, fy_3), G(gy_1, gy_2, fy_3), G(gy_1, gy_2, gy_3), \\ G(gy_1, fy_1, fy_1), G(gy_2, fy_2, fy_2), G(gy_3, fy_3, fy_3)\}.$$

If f and g are compatible, g-compatible, f-compatible, or compatible of type (P), they share a single common fixed point..

Proof. Let α0 be arbitrary. Since fY ⊆ gY , we can construct a sequence {βn} in Y such that βn = fαn = gαn+1 for n = 0, 1, 2, . . .

Assume that G(βn+1, βn+1, βn) > 0 for all n = 0, 1, 2, . . ., otherwise we obtain βn+1 = βn. Therefore βn+1 6= βn for all n = 0, 1, 2, . . .

Now, to prove {βn} is G-Cauchy in Y , consider

$$\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy = \int_0^{G(f\alpha_{n+1},f\alpha_{n+1},f\alpha_n)} \varphi(y)dy \leq \phi\left(\int_0^{L(\alpha_{n+1},\alpha_{n+1},\alpha_n)} \varphi(y)dy\right)$$ ..
.....(2)

where L(αn+1, αn+1, αn) = max{G(βn, βn+1, βn), G(βn, βn, βn−1), G(βn+1, βn+1, βn)}. If L(αn+1, αn+1, αn) = G(βn+1, βn+1, βn), then from (4.2) we have

$$\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy \leq \phi\left(\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy\right) < \int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy,$$

a contradiction. If L(αn+1, αn+1, αn) = G(βn, βn, βn−1), then from (4.2) we have

$$\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy \leq \phi\left(\int_0^{G(\beta_n,\beta_n,\beta_{n-1})} \varphi(y)dy\right) \leq \ldots \leq \phi^n\left(\int_0^{G(\beta_1,\beta_1,\beta_0)} \varphi(y)dy\right).$$

Let ε > 0 be given. Since φ n $\left(\int_0^{G(\beta_1,\beta_1,\beta_0)} \varphi(y)dy\right) \to 0$ as n → ∞, there exists an integer l_0 such that φ n $\left(\int_0^{G(\beta_1,\beta_1,\beta_0)} \varphi(y)dy\right) < \frac{1}{3}[\varepsilon - \phi(\varepsilon)]$ ∀ n ≥ l0.

Therefore $\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy < \frac{1}{3}[\varepsilon - \phi(\varepsilon)]$ ∀ n ≥ l0

Similarly, If L(αn+1, αn+1, αn) = G(βn, βn+1, βn), then we obtain

$$\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy < \frac{1}{3}[\varepsilon - \phi(\varepsilon)]$$

∀ n ≥ l1.........(3)

Let l = max{l0, l1}. Then,

$$\int_0^{G(\beta_{n+1},\beta_{n+1},\beta_n)} \varphi(y)dy < \frac{1}{3}[\varepsilon - \phi(\varepsilon)]$$

∀ k ≥ n ≥ l.............................(4)

By induction on k, we prove (4). By using (4), we can see that it holds for k = n + 1. (3). Assume that (3) is true for k = m.

$$\int_0^{G(\beta_m,\beta_m,\beta_n)} \varphi(y)dy < \epsilon \qquad \forall\, m \geq n \geq l.$$ ....
.................................(5)

For k = m + 1, we have

$$\int_0^{G(\beta_{m+1},\beta_{m+1},\beta_n)} \varphi(y)dy \leq \int_0^{G(\beta_n,\beta_{n+1},\beta_{n+1})} \varphi(y)dy + \phi\left(\int_0^{L(\alpha_{n+1},\alpha_{m+1},\alpha_{m+1})} \varphi(y)dy\right)$$

...................................(5)

where

$$L(\alpha_{m+1},\alpha_{m+1},\alpha_{n+1}) = \max\{G(\beta_m,\beta_{m+1},\beta_{m+1}) + G(\beta_{m+1},\beta_{m+1},\beta_{n+1}),$$
$$G(\beta_n,\beta_{n+1},\beta_{n+1}), G(\beta_m,\beta_m,\beta_n) + G(\beta_n,\beta_n,\beta_{n+1})\}.$$

If L(αm+1, αm+1, αn+1) = G(βm, βm+1, βm+1)+G(βm+1, βm+1, βn+1) = f(βm, βm+1)(say), then from (4.6) we have

$$\int_0^{G(\beta_{m+1},\beta_{m+1},\beta_n)} \varphi(y)dy \leq \int_0^{G(\beta_n,\beta_{n+1},\beta_{n+1})} \varphi(y)dy + \phi\left(\int_0^{f(\beta_m,\beta_{m+1})} \varphi(y)dy\right)$$
$$< \frac{1}{3}[\varepsilon - \phi(\varepsilon)] + \frac{1}{3}[\varepsilon - \phi(\varepsilon)] + \phi(\varepsilon) = \varepsilon.$$

If L(αm+1, αm+1, αn+1) = G(βm, βm, βn) + G(βn, βn, βn+1) = f(βm, βn, βn+1)(say), then from (6) we have

$$\int_0^{G(\beta_{m+1},\beta_{m+1},\beta_n)} \varphi(y)dy \leq \int_0^{G(\beta_n,\beta_{n+1},\beta_{n+1})} \varphi(y)dy + \phi\left(\int_0^{f(\beta_m,\beta_n,\beta_{n+1})} \varphi(y)dy\right)$$
$$< \frac{1}{3}[\varepsilon - \phi(\varepsilon)] + \phi(\varepsilon) + \frac{2}{3}[\varepsilon - \phi(\varepsilon)] = \varepsilon.$$

If L(αm+1, αm+1, αn+1) = G(βn, βn+1, βn+1), then from (4.6) we have

$$\int_0^{G(\beta_{m+1},\beta_{m+1},\beta_n)} \varphi(y)dy \leq \int_0^{G(\beta_n,\beta_{n+1},\beta_{n+1})} \varphi(y)dy + \phi\left(\int_0^{G(\beta_n,\beta_{n+1},\beta_{n+1})} \varphi(y)dy\right)$$
$$< \frac{1}{3}[\varepsilon - \phi(\varepsilon)] + \frac{1}{3}[\varepsilon - \phi(\varepsilon)] = \frac{2}{3}[\varepsilon - \phi(\varepsilon)] < \varepsilon.$$

Therefore by induction on k, we conclude that (4.4) holds good for all k ≥ n ≥ l. Since ε is arbitrary, we have

$$\int_0^{G(\beta_m,\beta_m,\beta_n)} \varphi(y)dy \to 0 \text{ as } n, m \to \infty.$$

Therefore G(βm, βm, βn) → 0 as n, m → ∞. That is {βn} is G-Cauchy. As Y is G-complete, ∃ a point s ∈ Y such that

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} f\alpha_n = \lim_{n\to\infty} g\alpha_{n+1} = s.$$

Case I: Let f and g be compatible. Then $\lim_{n\to\infty} G(fg\alpha_n, gf\alpha_n, gf\alpha_n) = 0.$ f- weak reciprocal continuity of (f, g) implies fgαn → fs or ggαn → gs.

First, let fgαn → fs. Since fY ⊆ gY , ∃ a point r ∈ Y such that fs = gr.

Therefore fgαn → gr which implies gfαn → gr, by the compatibility.

CONCLUSION

Every item in fuzzy set theory has a "degree of membership" between [0, 1]. Because it is impossible to compute distance functions with inexact values using traditional metric space theory, Kramosil and Michalek proposed the innovative concept of "fuzzy metric space (FMS)" to solve this issue (1975). Some of the primary elements of this research effort include the study of numerous "fixed point" outcomes in the area of FMS, such as "intuitionistic fuzzy metric spaces ()" and "- fuzzy metric spaces (-FMS)". We employ the CLRg and CLRST qualities to relax numerous constraints such as continuity, range confinement, and subspace closure, among others. In this newly defined space, we also propose the idea of "modified intuitionistic - fuzzy metric space (MI -FMS)" and investigate common and linked fixed point theorems. Our "fuzzy fixed point" findings also have some applications in the realm of "dynamic programming." There are eight chapters in this thesis, followed by references and a list of publications. In the first chapter, we provide an overview of our study issue, as well as its significance and applications. We discuss some of the domains in which "fuzzy fixed point theorems" are used.

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