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A study on irrationality of some numbers

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INTRODUCTION

Suppose that α and β are positive irrational numbers. In this paper, we give the following criterion that $\frac{\alpha}{\beta}$ is an irrational number.

THEOREM : Suppose that φ and ψ are positive functions such that $\lim_{q\to\infty} \frac{\varphi(q)}{\psi(q)} = \infty$. Let k be an arbitary fixed positive number with $k \geq 1$, and suppose that $\frac{\psi(q)}{k^2\psi(kq)} = O(1)$ as $q \to \infty$. Let α and β be positive irrational numbers. Suppose that :

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(1) there exists a number q' = q'(α) such that $\left|\alpha - \frac{p}{q}\right| > \frac{\varphi(q)}{q}$ for all $(p,q) \in Z \times N$ with $q \geq q'$. and

(II) the inequality $\left|\beta - \frac{p}{q}\right| < \frac{\psi(q)}{q}$ has infinitely many solutions $(p,q) \in Z \times N$. Then $\frac{\alpha}{\beta}$ is an irrational number.

Proof : Let q be a sufficiently large integer to ensure the validity of the later argument. Without any loss of generality, we may assume that $\alpha < \beta$. Suppose that $\frac{\alpha}{\beta}$ is a rational number, so that $\frac{\alpha}{\beta} = \frac{P}{Q}$, where P and Q are positive integers. Now, let $\frac{p}{q}$ be a rational number such that $\left|\beta - \frac{p}{q}\right| < \frac{\psi(q)}{q}$. Since $\beta - \frac{p}{q} = \frac{P}{Q}\alpha - \frac{p}{q} = \frac{Q}{P}\left[\alpha - \frac{Pp}{Qq}\right]$. we find that $\left|\beta - \frac{p}{q}\right| > \left|\alpha - \frac{Pp}{Qq}\right| = \left|\alpha - \frac{p^*}{q^*}\right|$.

where p* = Pp and q* = Qq.

By the statement (I), we obtain

$$\left|\alpha - \frac{p^*}{q^*}\right| > \frac{\varphi(q^*)}{q^*}.$$

This implies that $\left|\beta - \frac{p}{q}\right| > \frac{\varphi(q^*)}{q^*}$. (1)

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and by the assumption, we obtain

$$\left|\beta - \frac{p}{q}\right| < \frac{\psi(q)}{q} = Q.\frac{\psi(q)}{\psi(Qq)}.\frac{\psi(q^*)}{q^*} = Q^3.\frac{\psi(q^*)}{q^*}.O(1) < \frac{\psi(q^*)}{q^*} \quad as \quad q \to \infty. \qquad (2)$$

The relation (2) contradicts (1), and the theorem is proved.

EXAMPLES

For $\alpha = e$, Okano [3] proved that $\varphi(q) = \frac{\log \log q}{3q \log \log q}$.

Example 1 : If β is a Liouville number, then e/β is an irrational number.

Proof : As we can put $\psi(q) = \frac{1}{q^2}$, we deduce that e/β is an irrational number.

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Example 2 : Let m and k be integers with $m \geq 0$ and $k \geq 2$. If $\beta = [a_0, a_1, a_2, a_3, ...] = [m, 1^k, 2^k, 3^k, ...]$, then e/β is an irrational number.

Proof : Let $\frac{p_n}{q_n}$ be the n^{th} convergent of β. Since $q_n = n^k q_{n-1} + q_{n-2} \leq (n^k + 1) q_{n-1} < (n+1)^k q_{n-1}$ for $n \geq 2$, we have $q_n < (n\_1)^k q_{n-1} < ... < ((n+1)!)^k$ for $n \geq 1$.

Hence, $\log q_n < k \sum_{j=1}^{n+1} \log j < k \int_1^{n+2} \log x dx = k((n+2)\log(n+2) - (n+1))$.

Accordingly, $a_{n+1} = (n+1)^k \log q_n$ for all sufficiently large n, then

$$\left| \beta - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{q_{n+1} q_n^2} < \frac{1}{q_n^2 \log q_n}$$

has ifinitely many solutions (q_n,q_n). Consequently, as we can put $\psi(q) = \frac{1}{q \log q}$' we deduce that $\frac{e}{\beta}$ is an irrational number.

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