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SOLUTIONS OF HIGHER ORDER HOMOGENEOUS LINEAR MATRIX DIFFERENTIAL EQUATIONS FOR CONSISTENT AND NON CONSISTENT INITIAL CONDITIONS

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Abstract:

In this article, we study a class of linear matrix differetial equations (regular case) of higher order whose coefficients are square constant matrices.By using matrix pencil theory and the Weierstrass cannonical form of the pencil we obtain formulas for the solutions and we show that the solution is unique for consistent initial conditions and infinite for non-consistent initial conditions.Moreover we provide some numerical examples. These kinds of systems are inherent in many physical and engineering phenomena.

1. Introduction

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Linear Matrix Differential Equations (LMDEs) are inherent in many physical, engineering, mechanical, and financial/actuarial models.Having in mind such applications, for instance in finance, we provide the well-known input-output Leondief model and its several important extensions, advice [3]. In this article, our long-term purpose is to study the solution of LMDEs of higher order (1.1) into the mainstream of matrix pencil theory. This effort is signifficant, since there are numerous applications. Thus, we consider

$$A_n X^{(n)}(t) + A_{n-1} X^{(n-1)}(t) + \dots + A_1 X'(t) + A_0 X(t) = \mathbb{O} \qquad (1.1)$$

where $A_i, i = 0, 1, \dots, n \in \mathcal{M}(m \times m; \mathbb{F})$, (i.e. the algebra of square matrices with elements in the field F) with $X \in \mathcal{C}^\infty(\mathbb{F}, \mathcal{M}(m \times 1; \mathbb{F}))$. For the sake of simplicity we set $\mathcal{M}_m = \mathcal{M}(m \times m; \mathbb{F})$ and $\mathcal{M}_{nm} = \mathcal{M}(n \times m; \mathbb{F})$. In the sequel we adopt the following notations

$$\begin{gathered} Y_1(t) = X(t), \\ Y_2(t) = X'(t), \\ \dots \\ Y_{n-1}(t) = X^{(n-1)}(t), \\ Y_n(t) = X^{(n-1)}(t). \end{gathered}$$

$$\begin{gathered} Y_1'(t) = X'(t) = Y_2(t), \\ Y_2'(t) = X''(t) = Y_3(t), \\ \dots \\ Y_{n-1}(t) = X^{(n-1)}(t) = Y_n(t), \\ A_n Y_n'(t) = A_n X^{(n)}(t) = -A_{n-1} Y_n(t) - \dots - A_1 Y_2(t) - A_0 Y_1(t). \end{gathered}$$

ISSN-2230-9659

Or in Matrix form

$$FY'(t) = GY(t) \qquad (1.2)$$

where $Y(t) = [Y_1^T(t)Y_2^T(t)\dots Y_n^T(t)]^T$ (where $(\ )^T$ is the transpose tensor) and the coefficient matrices F;G are given by

$$\mathbf{F} = \begin{bmatrix} I_m & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & I_m & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & I_m & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & A_n \end{bmatrix}, \mathbf{G} = \begin{bmatrix} \mathbb{O} & I_m & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & I_m & \dots & \mathbb{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \dots & I_m \\ -A_1 & -A_2 & -A_3 & \dots & -A_{n-1} \end{bmatrix} \qquad (1.3)$$

with corresponding dimension of F;G and Z(t), mn_mn and mn_1, respectively. Matrix pencil theory has been extensively used for the study of Linear Differential Equations (LDEs) with time invariant coefficients, see for instance [3], [7]-[9]. Systems of type (1.1) are more general, including the special case when An = In, where In is the identity matrix of Mn, since the well-known class of higher-order linear matrix differential equations of Apostol-Kolodner type is derived straightforwardly, see [1] for n = 2, [2] and [10].

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The paper is organized as follows: In Section 2 some notations and the necessary preliminary concepts from matrix pencil theory are presented. Section 3 contains the case that system (1.1) has consistent initial conditions. In Section 4 the non consistent initial condition case is fully discussed. In this case, the arbitrarily chosen initial conditions which have physical meaning for (regular) systems, in some sense, can be created or structurally changed at a fixed time $t = t_0$. Hence, it is derived that (1.1) should adopt a generalized solution, in the sense of Dirac -solutions.

2. Mathematical Background and Notation

This brief section introduces some preliminary concepts and definitions from matrix pencil theory, which are being used throughout the paper.Linear systems of type (1.1) are closely related to matrix pencil theory, since the algebraic geometric, and dynamic properties stem from the structure by the associated pencil $sF - G$.

Definition 2.1. Given $F, G \in \mathcal{M}_{nm}$ and an indeterminate $s \in \mathbb{F}$, the matrix pencil $sF - G$ is called regular when m = n and $\det(sF - G) \neq 0$. In any other case, the pencil will be called singular.

Definition 2.2. The pencil $sF - G$ is said to be strictly equivalent to the pencil $s\tilde{F} - \tilde{G}$ if and only if there exist nonsingular $P \in \mathcal{M}_n$ and $Q \in \mathcal{M}_m$ such as

$$P(sF - G)Q = s\tilde{F} - \tilde{G}.$$

In this article, we consider the case that pencil is regular. Thus, the strict equivalence relation can be defined rigorously on the set of regular pencils as follows.

ISSN-2230-9659

Here, we regard (2.1) as the set of pair of nonsingular elements of Mn

$$g := \{(P, Q) : P, Q \in \mathcal{M}_n,\ P, Q \text{ nonsingular}\} \qquad (2.1)$$

and a composition rule $*$ defined on g as follows:

$$* : g \times g \text{ such that } (P_1, Q_1) * (P_2, Q_2) := (P_1 \cdot P_2, Q_2 \cdot Q_1). \qquad (2.2)$$

It can be easily verified that $(g, *)$ forms a non-abelian group. Furthermore, an action $\circ$ of the group $(g, *)$ on the set of regular matrix pencils $\mathcal{L}_n^{\text{reg}}$ is defined as $\circ : g \times \mathcal{L}_n^{\text{reg}} \to \mathcal{L}_n^{\text{reg}}$ such that

$$((P, Q), sF - G) \to (P, Q) \circ (sF - G) := P(sF - G)Q.$$

This group has the following properties:

(a) $(P_1, Q_1) \circ [(P_2, Q_2) \circ (sF - G)] = (P_1, Q_1) * (P_2, Q_2) \circ (sF - G)$ for every nonsingular $P_1, P_2 \in \mathcal{M}_n$ and $Q_1, Q_2 \in \mathcal{M}_n$.

(b) $e_g \circ (sF - G) = sF - G$, $sF - G \in \mathcal{L}_n^{\text{reg}}$ where $e_g = (I_n, I_n)$ is the *identity element* of the group $(g, *)$ on the set of $\mathcal{L}_n^{\text{reg}}$ defines a transformation group

ISSN-2230-9659

$$g \circ (sF - G) := \{(P, Q) \circ (sF - G) : (P, Q) \in g\} \subseteq \mathcal{L}_n^{\text{reg}}$$

will be called the orbit of $sF - G$ at g. Also N defines an equivalence relation on

$\mathcal{L}_n^{\text{reg}}$ which is called a strict-equivalence relation and is denoted by $\mathcal{E}_{s-e}$. So, $(sF - G)\mathcal{E}_{s-e}(s\tilde{F} - \tilde{G})$ if and only if $P(sF - G)Q = s\tilde{F} - \tilde{G}$, where $P, Q \in \mathcal{M}_n$ are nonsingular elements of algebra $\mathcal{M}_n$. The class of $\mathcal{E}_{s-e}(sF - G)$ is characterized by a uniquely defined element, known as a complexWeierstrass canonical form, $sF_w - Q_w$, , see [6], specified by the complete set of invariants of $\mathcal{E}_{s-e}(sF - G)$. This is the set of elementary divisors (e.d.) obtained by factorizing the invariant polynomials $f_i(s, \hat{s})$ into powers of homogeneous polynomials irreducible over field F. In the case where $sF - G$ is a regular, we have e.d. of the following type:

- e.d. of the type s^p are called zero finite elementary divisors (z. f.e.d.)
- e.d. of the type $(s - a)^\pi, a \neq 0$ are called nonzero finite elementary divisors (nz.f.e.d.)
- e.d. of the type ^sq are called in_nite elementary divisors (i.e.d.). Let B1;B2; : : : ;Bn be elements of $\mathcal{M}_n$. The direct sum of them denoted by B1 _ B2 _ _ _ _ _ Bn is the block diagfB1;B2; : : : ;Bng.

Then, the complex Weierstrass form $sF_w - Q_w$ of the regular pencil $sF - G$ is defined by $sF_w - Q_w := sI_p - J_p \oplus sH_q - I_q$, where the _rst normal Jordan type element is uniquely defined by the set of f.e.d.

ISSN-2230-9659

$$(s-a_1)^{p_1}, \ldots, (s-a_\nu)^{p_\nu}, \quad \sum_{j=1}^{\nu} p_j = p \tag{2.3}$$

of $sF - G$ and has the form

$$sI_p - J_p := sI_{p_1} - J_{p_1}(a_1) \oplus \cdots \oplus sI_{p_\nu} - J_{p_\nu}(a_\nu). \tag{2.4}$$

And also the q blocks of the second uniquely defined block $sH_q - I_q$ correspond to the i.e.d.

$$\hat{s}^{q_1}, \ldots, \hat{s}^{q_\sigma}, \quad \sum_{j=1}^{\sigma} q_j = q \tag{2.5}$$

$$sH_q - I_q := sH_{q_1} - I_{q_1} \oplus \cdots \oplus sH_{q_\sigma} - I_{q_\sigma}. \tag{2.6}$$

Thus, H_q is a nilpotent element of $\mathcal{M}_n$ with index $\tilde{q} = \max\{q_j : j = 1, 2, \ldots, \sigma\}$, where

$$H_q^{\tilde{q}} = \mathbb{O},$$

and $I_{p_j}, J_{p_j}(a_j), H_{q_j}$ are defined as

$$I_{p_j} = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \in \mathcal{M}_{p_j}, \quad J_{p_j}(a_j) = \begin{bmatrix} a_j & 1 & 0 & \ldots & 0 \\ 0 & a_j & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & a_j & 1 \\ 0 & 0 & 0 & 0 & a_j \end{bmatrix} \in \mathcal{M}_{p_j}$$

$$H_{q_j} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{q_j}. \tag{2.7}$$

In the last part of this section, some elements for the analytic computation of $e^{A(t-t_0)}, t \in [t_0, \infty)$ are provided. To perform this computation, many theoretical and numerical methods have been developed. Thus, the interesting readers might consult papers [2, 4, 10, 11, 13] and the references therein. In order to have computational formulas, see the following Sections 3 and 4, the following known results should firstly be mentioned.

ISSN-2230-9659

Lemma 2.3 ([4]). $e^{J_{p_j}(a_j)(t-t_0)} = (d_{k_1k_2})_{p_j}$, *where*

$$d_{k_1k_2} = \begin{cases} e^{a_j(t-t_0)}\dfrac{(t-t_0)^{k_2-k_1}}{(k_2-k_1)!}, & 1 \le k_1 \le k_2 \le p_j \\ 0, & \text{otherwise} \end{cases}$$

Another expression for the exponential matrix of Jordan block, see (2.7), is provided by the following Lemma.

Lemma 2.4 ([13]).

$$e^{J_{p_j}(a_j)(t-t_0)} = \sum_{i=0}^{p_j-1} f_i(t-t_0)[J_{p_j}(a_j)]^i \tag{2.8}$$

where the $f_i(t-t_0)$*'s are given analytically by the following* p_j *equations:*

$$f_{p_j-1-k}(t-t_0) = e^{a_j(t-t_0)}\sum_{i=0}^{k} b_{k,i}a_j^{k-i}\frac{(t-t_0)^{p_j-1-i}}{(p_j-1-i)!}, \quad k = 0,1,2,\ldots,p_j-1 \tag{2.9}$$

where

$$b_{k,i} = \sum_{l=0}^{k-i}\binom{p_j}{l}\binom{k-l}{i}(-1)^l$$

and

$$[J_{p_j}(a_j)]^i = (c^{(i)}_{k_1k_2})_{p_j}, \text{ for } 1 \le k_1, k_2 \le p_j \tag{2.10}$$

$$c^{(i)}_{k_1k_2} = \binom{i}{k_2-k_1}a_j^{i-(k_2-k_1)}.$$

In this section, the main results for consistent initial conditions are analytically presented for the regular case. The whole discussion extends the existing literature; see for instance [2]. Moreover, it should be stressed out that these results offer the necessary mathematical framework for interesting applications, see also introduction. Now, in order to obtain a unique solution, we deal with consistent initial value problem. More analytically, we consider the system

ISSN-2230-9659

$$A_n X^{(n)}(t) + A_{n-1} X^{(n-1)}(t) + \ldots + A_1 X'(t) + A_0 X(t) = \mathbb{O} \qquad (3.1)$$

with known initial conditions

$$X(t_0), X'(t_0), \ldots, X^{(n-1)}(t_0). \qquad (3.2)$$

analytically, we consider the system

$$FY'(t) = GY(t), Y(t_0) \qquad (3.3)$$

From the regularity of $sF - G$, there exist nonsingular $\mathcal{M}(mn \times mn, \mathbb{F})$ matrices P and Q such that (see also section 2), such as

$$PFQ = F_w = I_p \oplus H_q, \qquad (3.4)$$

$$PGQ = G_w = J_p \oplus I_q, \qquad (3.5)$$

where I_p, J_p, H_q and I_q are given by (2.7) where

$$I_p = I_{p_1} \oplus \ldots \oplus I_{p_\nu},$$
$$J_p = J_{p_1}(a_1) \oplus \ldots \oplus J_{p_\nu}(a_\nu),$$
$$H_q = H_{q_1} \oplus \ldots \oplus H_{q_\sigma},$$
$$I_q = I_{q_1} \oplus \ldots \oplus I_{q_\sigma}.$$

Note that $\sum_{j=1}^{\nu} p_j = p$ and $\sum_{j=1}^{\sigma} q_j = q$, where $p + q = n$.

Lemma 3.1. *System (3.1) is divided into two subsystems: The so-called slow subsystem*

$$Z_p'(t) = J_p Z_p(t), \qquad (3.6)$$

and the relative fast subsystem

$$H_q Z_q'(t) = Z_q(t). \qquad (3.7)$$

Proof. Consider the transformation

$$Y(t) = QZ(t). \qquad (3.8)$$

Substituting the previous expression into (3.1) we obtain

$$FQZ'(t) = GQZ(t).$$

Whereby, multiplying by P, we arrive at

$$F_w Z(t) = G_w Z(t).$$

Moreover, we can write $Z(t)$ as $Z(t) = \begin{bmatrix} Z_p(t) \\ Z_q(t) \end{bmatrix}$. Taking into account the above expressions, we arrive easily at (3.2) and (3.7). □

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Proposition 3.2. *The subsystem has the unique solution*

$$Z_p(t) = e^{J_p(t-t_0)} Z_p(t_0), t \geq t_0, \tag{3.9}$$

where $\sum_{j=1}^{v} p_j = p$.

Proof. See [9],[13] □

Proposition 3.3. *The fast subsystem (3.7) has only the zero solution.*

Proof. Let q_* be the index of the nilpotent matrix H_q, i.e. $H_q^{q_*} = \mathbb{O}$), we obtain the following equations

$$H_q Z_q'(t) = Z_q(t)$$
$$H_q L Z_q'(t) = L Z_q(t)$$
$$H_q[sY(s) - Z_q(t_0)] = Y(s)$$
$$(sH_q - I_q)Y(s) = H_q Z_q$$

Where $Y(s) = LZ_q(t) = \int_{t_0}^{\infty} Z_q(t) e^{-s(t-t_0)} dt$ is by definition the Laplace transform of Z_q.It is easy to show that $det(sH_q - I_q) \neq 0$ and that $(sH_q - I_q)^{-1} = -\sum_{n=0}^{q_{max}-1} (sH_q)^n$, while $H_q^n = 0$ for $n \geq q_*$

$$Y(s) = (sH_q - I_q)^{-1} H_q Z_q$$
$$Y(s) = -\sum_{n=0}^{q_*-1} (sH_q)^n H_q Z_q = -\sum_{n=0}^{q_*-1} s^n H_q^{n+1} Z_q$$
$$Y(s) = -\sum_{n=1}^{q_*-2} (s)^{n-1} H_q^n Z_q$$
$$L^{-1}Y(s) = -\sum_{n=1}^{q_*-2} L^{-1} s^{n-1} H_q^n Z_q(t_0)$$
$$Z_q = -\sum_{n=1}^{q_*-2} \delta^n (t - t_0) H_q^n Z_q(t_0)$$
$$Z_q = 0$$

where $\delta(t - t_0)$ is by definition the Dirac function

$$\int_{-\infty}^{\infty} \delta(t - t_0) dt = 1, t = t_0$$
$$\delta(t - t_0) = 0, t \neq t_0$$

The conclusion, i.e. $Y_q(t) = \mathbb{O}$, is obtained by repetitively substitution of each equation in the next one, and using the fact that $H_q^{q^*} = \mathbb{O}$.

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Theorem 3.4. Consider the system (3.1-3.2).Then the solution is unique if and only if the initial conditions are consistent.Moreover the analytic solution of (3.1-3.2) is given by

$$X(t) = Q_p^1 e^{J_p(t-t_0)} Z_p(t_0) \qquad (3.10)$$

Proof. Let $Q = [Q_p Q_q]$ where $\mathbf{Q_p} \in \mathcal{M}_{(mn)p}$ and $\mathbf{Q_q} \in \mathcal{M}_{(mn)q}$;Combining (3.8) and (??), we obtain

$$Y(t) = QZ(t) = [Q_p\ Q_q] \begin{bmatrix} Z_p(t) \\ \mathbb{O} \end{bmatrix} = Q_p e^{J_p(t-t_0)} Z_p(t_0).$$

solution that exists if and only if $Y(t_0) = Q_p Z_p(t_0)$ or $Y(t_0) \in colspan Q_p$. The columns of Qp are the p eigenvectors of the finite elementary divisors (eigenvalues) of the pencil sA-B.Let In that case the system has the unique solution

$$X(t) = Q_p^1 e^{J_p(t-t_0)} Z_p(t_0) \qquad (3.11)$$

where Q_p^1 is defined as

$$Q_p = \begin{bmatrix} Q_p^1 \\ Q_p^2 \end{bmatrix}.$$

and $\mathbf{Q_p^1} \in \mathcal{M}_{pp}$ □

ISSN-2230-9659

Proposition 4.1. Consider the system (3.3).Then for non consistent initial conditions $(Y(t_0) \notin colspanQ_p)$the system has infinite solutions. Proof. Let Q_p,Q_q be the matrices defined in Theorem 3.5.If the initial conditions are non consistent then $Y(t_0) \notin \text{colspan}Q_p$ and $Z_q(t_0) \neq 0$.Moreover $Y(t_0) = Q_pZ_p(t_0) + Q_qZ_q(t_0)$. This means (3.3) is defined for t 6= t0 because if $t = t_0$ then $FY'(t_0) = GY(t_0)$and $Z_q(t_0) = \mathbb{O}$which is a contradiction.Let $H(t - t_0)$ be the Heaviside function and

$$f(t) = H(t - t_0) - H(t_0 - t) = \begin{cases} 1 & , \quad t > t_0 \\ 0 & , \quad t = t_0 \end{cases}$$

$$g(t) = H(t_0 - t) = \begin{cases} 1 & , \quad t = t_0 \\ 0 & , \quad t \neq t_0 \end{cases}$$

Then the system can be written as

$$f(t)FY'(t) = GY(t) - g(t)GY(t_0), t \geq t_0 \qquad (4.1)$$

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Let

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} X'''(t) + \begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix} X''(t) + \begin{bmatrix}-2 & 3\\ 1 & 1\end{bmatrix} X'(t) + \begin{bmatrix}4 & -2\\ -1 & -1\end{bmatrix} X(t) = \mathbb{O} \quad (5.1)$$

Where $X(t) = [X_1(t)^T X_2(t)^T]^T$.We adopt the following notations

$$Y_1(t) = X(t),$$
$$Y_2(t) = X'(t),$$
$$Y_3(t) = X''(t).$$

$$Y_1'(t) = X'(t) = Y_2(t),$$
$$Y_2'(t) = X''(t) = Y_3(t),$$
$$A_3Y_3'(t) = A_3X'''(t) = -A_2Y_3(t) - A_1Y_2(t) - A_0Y_1(t).$$

Or in Matrix form

$$FY'(t) = GY(t) \quad (5.2)$$

where $Y(t) = [Y_1^T(t)Y_2^T(t)Y_3^T(t)]^T$ (where $(\ )^T$ is the transpose tensor) and the coefficient matrices F, G are given by

$$\mathbf{F} = \begin{bmatrix} I_2 & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & I_2 & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & A_n\end{bmatrix}, \mathbf{G} = \begin{bmatrix}\mathbb{O} & I_3 & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & I_3\\ -A_1 & -A_2 & -A_3\end{bmatrix} \quad (5.3)$$

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and

$$sF-G=s\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&1\\0&0&0&0&0&0\end{bmatrix}-\begin{bmatrix}0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\-4&2&2&-3&-2&-1\\1&1&-1&-1&0&0\end{bmatrix}$$

s-1,s-2,s-3 finite elementary divisors and $\bar{s}^3$ the infinte elementary divisor of degree 3 of the pencil sF-G.There exist matrices non singular P,Q such that PAQ=F_w and PGQ=G_w.Where

$$\mathbf{F_w}=\begin{bmatrix}I_3&\mathbb{O}\\\mathbb{O}&H_3\end{bmatrix},\mathbf{G_w}=\begin{bmatrix}J_3&\mathbb{O}\\\mathbb{O}&I_3\end{bmatrix}\tag{5.4}$$

Let $Y(t)=QZ(t)$ then

$$FY'(t)=GY(t)$$
$$PFQZ'(t)=PGQZ(t)$$
$$F_wZ'(t)=G_wZ(t)$$

For cosistent inital conditions the solution is

$$Y(t)=Q_3^1e^{J_3(t-t_0)}Z_3(t_0)X(t)=Q_3^1e^{J_3(t-t_0)}Z_3(t_0)\tag{5.5}$$

and for non conistent initial conditions the solution is

$$X(t)=f(t)Q_3e^{J_3(t-t_0)}C+g(t)\sum_{i=0}^{2}\frac{(t-t_0)^i}{i!}X^{(i)}(t_0),\tag{5.6}$$

where

$$J_3=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$$

and

$$e^{J_3t}=\begin{bmatrix}e^t&0&0\\0&e^{2t}&0\\0&0&e^{3t}\end{bmatrix}$$

The columns of Q_p are the eigenvectors of the eigenvalues 1,2,3

$$Q_p^T=\begin{bmatrix}3&-5&3&-5&3&-5\\1&-1&2&-2&4&-4\\1&-1&3&-3&9&-9\end{bmatrix}^T$$

Let the initial values of the system be

$$X(0)=\begin{bmatrix}1\\-3\end{bmatrix},X'(0)=\begin{bmatrix}-2\\0\end{bmatrix},X''(0)=\begin{bmatrix}-10\\8\end{bmatrix}$$

and

$$Y(0)^T=\begin{bmatrix}1&-3&-2&0&-10&8\end{bmatrix}^T$$

Then $Y(0)\in$ colspanQ_p (consistent initial conditions) and the solution of the system is

$$Y(t)=Q_3e^{J_3t}Z_3(0)\tag{5.7}$$

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$$Y(0) = Q_3 Z_3(0)$$
$$Z_3(0)^T = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T$$

and the solution of the system is

$$Y(t) = Q_3 e^{J_3 t} Z_3(0)$$
$$Y(t) = \begin{bmatrix} 3e^t - e^{2t} - e^{3t} \\ -5e^t + e^{2t} + e^{3t} \\ 3e^t - 2e^{2t} - 3e^{3t} \\ -5e^t + 2e^{2t} + 3e^{3t} \\ 3e^t - 4e^{2t} - 9e^{2t} \\ -5e^t + 4e^{2t} + 9e^{3t} \end{bmatrix}$$

$$X(t) = \begin{bmatrix} 3e^t - e^{2t} - e^{3t} \\ -5e^t + e^{2t} + e^{3t} \end{bmatrix}$$

Next assume the initial conditions

$$X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, X''(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and

$$Y(0)^T = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}^T$$

Then $Y(0) \notin \text{colspan} Q_p$ non consistent initial conditions and the solution is

$$X(t) = f(t) Q_3^1 e^{J_3 t} C + g(t) \sum_{i=0}^{2} \frac{t_i}{i!} X^{(i)}(0)$$
$$X(t) = f(t) \begin{bmatrix} 3e^t c_1 + e^{2t} c_2 + e^{3t} c_3 \\ -5e^t c_1 - e^{2t} c_2 - e^{3t} c_3 \end{bmatrix} + g(t) \frac{t^2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \geq 0$$

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Conclusions

In this article, we study the class of linear rectangular matrix differential equations of higher-order whose coefficients are square constant matrices. By taking into consideration that the relevant pencil is regular, we get effected by the Weierstrass canonical form in order to decompose differential system into two sub-systems (i.e. the slow and the fast sub-system). Afterwards, we provide analytical formulas for that general class of Apostol-Kolodner type of equations when we have consistent and non-consistent initial conditions. Moreover, as a further extension of the present paper, we can discuss the case where the pencil is singular. Thus, the Kronecker canonical form is required. The non-homogeneous case has also a special interest, since it appears often in applications. For all these, there is some research in progress.

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