Existence Results for a System of Fractional Differential Equations with Fractional Order Random Time Scale

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Abstract – This paper explores the extant of unique solution for a set of non-linear fractional differential equation with fractional order in capricious time lamina. The solutions are demonstrated by some basic fixed-point theory, Kooi's, Rogers and Krasnoselskii-Krein conditions

Keywords – Fractional Impulsive Differential Equations, Initial Value Problem, Fixed Point Theorem.

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INTRODUCTION

Consider the fractional differential equations with initial conditions as given below

$$\left.\begin{array}{ll} {}^{S}_{s}D^{a}u(s)=h(s,u(s)) & s\in[s_0,s_0+\alpha],0<a\leq 1 \\ {}^{S}_{s}Iu(s_0)=0 & \end{array}\right\} \quad (1.1)$$

where ${}^{S}_{s}D^{a}$ is the S time scale Riemann Liouville fractional derivative of order a.

${}^{S}_{s}I$ the fractional integral of Riemann-Liouville and $[s_0,s_0+\alpha]_S$ is an arbitrary interval on *S*.

As a part of theoretical & potential applications, theory of time lamina calculus is involved to concern together difference and differential equation [1]. suppose that $h(s)$ is a continuous function with right dense. Many authors have tried and proved the life and one-of-a-kindness of the first order differential equations with initial and boundary time scales conditions by various methods and criteria.

The idea of this content arises from reference papers [10-24] in which Krasnoselskii-Krein and Nagumo conditions on non-linear term *h,* excluding Lipschitz assumption are exposed to derive the main results.

Consider the first of the following orders ordinary differential equation with two classes

$$\left.\begin{array}{ll} u^{\Delta}(s)=h(s,u(s)) & s\in[s_0,s_0+\alpha]_S \\ u(s_0)=0 & \end{array}\right\} \quad (1.2)$$

and fractional differential equation with fractional order

$$\left.\begin{array}{ll} {}^{S}_{s}D^{a}u(s)=h(s,u(s)) \;\; s\in[s_0,s_0+\alpha]_S & 0<a\leq 1 \\ {}^{S}_{s}I^{1-a}u(s_0)=0 & \end{array}\right\} \quad (1.3)$$

In section 2, few definitions and fundamental statements are added in such a way to prove main results.

In section 3, the main theorem is illustrated. We first set up unique solution for first order problem under Krasnoselskii-Krein conditions. Then we extend the proof to successive approximation, which converge to unique solution.

2. PRELIMINARIES

We recollect basic consequence and definition from time lamina calculus.

A chronometer Let card (S)>=2 is a non-empty closed subset of S. The forward and backward jump operators $\xi,\eta:S\rightarrow S$ are respectively defined by

$$\xi(s)=\inf\{t\in S:t>s\}$$

&

$$\eta(s)=\sup\{t\in S:t<s\} \qquad (2.4)$$

The point $s\in S$ is defined as follows

$$\eta(s)=s;\ \text{left dense} \qquad \eta(s)<s;\ \text{left scattered}$$
$$\eta(s)=s;\ \text{right dense} \qquad \eta(s)>s;\ \text{right scattered}$$

Let

$$\begin{cases} S^{\kappa}=S\setminus\{\max S\};\ \text{when } S \text{ admits a left scattered maximum} \\ S^{\kappa}=S \qquad ;\ \text{otherwise}\end{cases}$$

Denote $A_s=A\cap S.$ I_S is interval of $S,$ where I is an interval of $R.$

Definition 2.1. Delta Derivative [1]

Assume $h:S\to R$ and let $s\in S^{\kappa}.$ Define

$$h^{\Delta}(s)=\lim_{t\to s}\frac{h(\xi(t))-h(s)}{\xi(t)-s};\quad s\neq\xi(t) \qquad (2.5)$$

provided the limit exist. Here $h^{\Delta}(s)$ is called delta derivative of h at $s.$ Also, h is referred as delta differentiable on S^{κ} provided h^{Δ} exists for all $s\in s^{\kappa}.$ The function $h^{\Delta}:s^{\kappa}\to R$ is called the delta derivative of h on $s^{\kappa}.$

Definition 2.2. [6]

A function $h:S\to R$ Only if it is rd-continuous is it considered rd-continuous right dense point continuous in S and its left sided limits exists at left dense points in $S.$ C_{rd} denotes a Banach space with norm and a set of rd-continuous functions. Similarly, a function $h:S\to R$ is called ld-continuous only if it is continuous at left dense point in S. The set of ld-continuous function $h:S\to R$ is represented by C_{ld}. For $h\in C_{rd}$, define $\|h\|=Sup_{rd}|h(s)|.$

Definition 2.3. Delta antiderivative [6]

A function $H:[\alpha,\beta]_s\to R$ A function's delta antiderivative is referred to as a function's delta antiderivative. $H:[\alpha,\beta)_s\to R$ provided H is continuous on $[\alpha,\beta]_s,$ delta-differentiable on $[\alpha,\beta)_s$ and $H^{\delta}(s)=h(s)$ for all $s\in[\alpha,\beta)_s.$ Then we define the $\Delta-$ integral oh h from α to β by

$$\int_{\alpha}^{\beta}h(s)\Delta s\overset{\Delta}{=}H(\beta)-H(\alpha) \qquad (2.6)$$

Definition 2.4. Fractional integral on time scales [6]

Suppose S is a time scale, $[\alpha,\beta]$ is an interval of S_1 & f is an integrable function on $[\alpha,\beta].$ Let $0<a<1.$ Then the left fractional integral of order a of f is defined by

$${}_{\alpha}^{S}I_s^a f(s)=\int_{\alpha}^{s}\frac{(s-t)^{a-1}}{\Gamma(a)}f(t)\Delta t\ ; \qquad (2.7)$$

where Γ is a gamma function

Definition 2.5. [Fractional Riemann Liouville Derivative on time Scale]

Let S be a time scale, $s\in S,0<a<1,$ and $f:S\to R.$ And there was the left. Fractional derivative of order Riemann-Liouville a of f is defined by

$${}_{\alpha}^{S}D_s^a f(s)=\frac{1}{\Gamma(1-a)}\left[\int_{\alpha}^{s}(s-t)^{-a}f(t)\Delta t\right]^{\Delta} \qquad (2.8)$$

We can use ${}_{s}^{S}I^{a}$ instead of ${}_{s_0}^{S}I_s^{a}$ and ${}_{s}^{S}I^{a}$ instead of ${}_{s_0}^{S}D_s^{a}$ when $\alpha=s_0.$

Lemma 2.1. Let h be a non-decreasing continuous function on the $[\alpha,\beta]_s.$ We define extension h of h to the non-imaginary interval $[\alpha,\beta]$ by

$$\bar{h}(t)=\begin{cases}h(t)\ if\ t\in S\\ 0\quad if\ t\in(s,\xi(s))\notin S\end{cases} \qquad (2.9)$$

$$\text{Then}\ \int_{\alpha}^{\beta}h(s)\Delta s\leq\int_{\alpha}^{\beta}\bar{h}(s)\Delta s \qquad (2.10)$$

$\bar{h}^{\Delta}(s)=h^{\Delta}(s),$ for every $s\in(\alpha,\beta)_s.$

Lemma 2.2. [5] Let $x:[s_0,s_0+\alpha]_s\to R$ be continuous. Then the general solution of the differential equation $v^{\Delta}(s)=x(s)$ is given by (2.11)

$$v(s)=v(s_0)+\int_{s_0}^{s}x(t)\Delta t,\quad s\in[s_0,s_0+\alpha] \qquad (2.12)$$

Lemma 2.3. [6] For any function h integrable on $[s_0,s_0+\alpha]_s$, we have the following

$$\left({}_{s}^{S}D^{a}\cdot{}_{s}^{S}I^{a}\right)(h)=h \qquad (2.13)$$

Lemma 2.4. [6] Let $h \in C([s_0, s_0+\alpha]_S)$ & $0<a<1$. If ${}_s^S I^{1-a} h(s)|_{s=s_0} = 0,$ then

$$\left({}_s^S I^a \cdot {}_s^S D^a\right)(h) = h. \qquad (2.14)$$

Lemma 2.5. [6] Let $0<a<1$ and $h:[s_0, s_0+\alpha]_S \times R \to R.$ The function v is a solution of problem (1.2) if and only if it is a solution of the following integral equation

$$v(s) = \frac{1}{\Gamma(a)} \int_{s_0}^{s} (s-t)^{a-1} h(t, v(t)) \Delta t \quad s \in [s_0, s_0+\alpha]_S \qquad (2.15)$$

Lemma 2.6. [22] The of the equation

$${}_{RL}D_{s_0}^{a} R(s) = [R(s)]^{\delta} \qquad (2.16)$$

is given by

$$R(s) = L(s-s_0)^{\xi} \qquad (2.17)$$

where $L = (\Gamma(1-a))^{\frac{1}{1-\delta}}$ and $\xi = \frac{1}{1-\delta}$ & ${}_{RL}D_{s_0}^{a}$ is the fractional Riemann-Liouville derivative of order $a \in (0,1)$ on the interval $[s_0, s_0+\alpha].$

3. MAIN RESULTS

To prove the main result, define

$$T_0 = \{(s,y): \quad s \in [s_0, s_0+\alpha], \quad |y| \le \beta, \quad \alpha, \beta \in R^+\}.$$

3.1. Results of Uniqueness for first order Ordinary Differential Equation:

Theorem 3.1.1. (Conditions of Krasnoselskii-Krein)

Let $h(s,y)$ be non-discontinuous in T_0 and for all $(s,y), (s,\bar{y}) \in T_0$ satisfying

$$\text{(A1)} \quad |h(s,y) - h(s,\bar{y})| \le k|s-s_0|^{-1} |y-\bar{y}|, \; s \ne s_0$$

$$\text{(A2)} \quad |h(s,y) - h(s,\bar{y})| \le c|s-s_0|^{-1} |y-\bar{y}|^{\delta},$$

for some positive constants *c* and *k;* also the non-imaginary number δ which lies between 0 and 1 such that $k(1-\delta)<1.$ Then, the first order initial value problem (1.2) has only one solution on $[s_0, s_0+\alpha]_S.$

Proof:

Suppose *p* and *q* are two solutions of (1.2) in $[s_0, s_0+\alpha]_S.$ We have to prove that $p \equiv q.$

Let us define $\psi(s)$ and $Q(s)$ by

$$\left.\begin{aligned} &\psi(s) = |p(s) - q(s)|, \text{ for every } s \in [s_0, s_0+\alpha]_S \\ &Q(s) = \int_{s_0}^{s} c\,\psi^{-\delta}(t)\,dt \text{ for every } s \in [s_0, s_0+\alpha] \end{aligned}\right\} \qquad (3.18)$$

Such that $\bar{\psi}$ is the extension of ψ to the real interval $[s_0, s_0+\alpha].$ From condition (A2) that

$$\begin{aligned} \psi(s) &= \left|\int_{s_0}^{s} [h(t,p(t)) - h(t,q(t))] \Delta t\right| \\ &\le \int_{s_0}^{s} |h(t,p(t)) - h(t,q(t))| \Delta t \qquad (3.19) \\ &\le \int_{s_0}^{s} c|p(t)-q(t)|^{\delta} \Delta t \le \int_{s_0}^{s} c|\bar{p}(t)-\bar{q}(t)|^{\delta} dt = Q(t) \end{aligned}$$

Consequently, since $Q(s_0)=0,\ Q(s)>0 \text{ for } s>s_0, \text{and } Q^{\Delta}(s) = c\psi^{-\delta}(s),$ for every $s \in [s_0, s_0+\alpha]_S.$ It is concluded from (3.18) and (3.19) that $Q'(s) \le cQ^{\delta}(s),$ for every

$$s \in [s_0, s_0+\alpha]. \qquad (3.20)$$

That is $\int (1-\delta) Q^{(1-\delta)}(s) Q'(s) ds \le \int c(1-\delta) Q^{(1-\delta)} Q^{\delta}(s) ds$. It is reduced to

$$Q^{(1-\delta)}(s) \le c(1-\delta)(s-s_0) \qquad (3.21)$$

Hence

$$\psi(s) \le c^{(1-\delta)^{-1}} (1-\delta)^{(1-\delta)^{-1}} (s-s_0)^{(1-\delta)^{-1}} \qquad (3.22)$$

$$\text{Denote } \phi(s) = \frac{\psi(s)}{(s-s_0)^k} \Rightarrow 0 \le \phi(s) \le \frac{(s-s_0)^{(1-\delta)^{-1}-k}}{c^{(1-\delta)^{-1}}(1-\delta)^{(1-\delta)^{-1}}} \text{ for every } s \in [s_0, s_0+\alpha]_S \qquad (3.23)$$

That is the exponent of s in the above constraint is non-negative, since $\frac{1}{k(1-\delta)} > 1$.

Hence $\lim_{s \to s_0} \phi(s) = 0.$ Therefore if we define $\phi(s_0)=0,$ then the function is rd-continuous in $[s_0, s_0+\alpha]_S.$ To prove $\psi = 0$ on $[s_0, s_0+\alpha]_S.$ Assume that ϕ does not disappear at some points *s;* that is $\phi(s)>0$ on $[s_0, s_0+\alpha].$ Then there arise a maximum $n>0,$ when s equals to some $s_1 : s_0 < s_1 < s_0+\alpha$ such that $\phi(t) < n < \phi(s_1), \text{ for } t \in [s_0, s_1)_S.$ From condition (A1), we have

$$\begin{aligned} n = \phi(s_1) &= (s_1-s_0)^{-k} \psi(s_1) \\ &\le (s_1-s_0)^{-k} \int_{s_0}^{s_1} k(t-s_0)^{-1} \psi(t) \Delta t \le (s_1-s_0)^{-k} \int_{s_0}^{s_1} k(t-s_0)^{k-1} \phi(t) \Delta t \qquad (3.24) \\ &< n(s_1-s_0)^{-k} \int_{s_0}^{s_1} k(t-s_0)^{k-1} \Delta t < (s_1-s_0)^{-k} \int_{s_0}^{s_1} k(t-s_0)^{k-1} dt < n, \end{aligned}$$

which is a contradiction. Hence, there exist unique solution.

Theorem 3.1.2. Kooi's Condition

Let $h(s,y)$ be non-discontinuous in T_0 and for all $(s,y), (s,\bar{y}) \in T_0$ satisfying

(B1) $|h(s,y)-h(s,\bar{y})| \le k|s-s_0|^{-1}|y-\bar{y}|,\ s \ne s_0$

(B2) $|s-s_0|^{b}|h(s,y)-h(s,\bar{y})| \le c\,|y-\bar{y}|^{\delta}$, for some positive constants c and k from real line.

Also, real numbers b,δ are defined as $0<b<\delta<1$, and $k(1-\delta)<1-b.$ Then the first order initial value problem (1.2) has only one solution on $[s_0,s_0+\alpha]_S.$

Proof. Similar procedure from theorem 3.1.1 is followed here to prove the given statement.

3.2. Existence of Solution on Time Lamina by Krasnoselskii-Krein Conditions

Theorem 3.2.3. Assume that conditions (A1) and (A2) are satisfied, then the consecutive estimations given by

$$p_{m+1}(s)=\int_{s_0}^{s} h(t,p_m(t))\Delta t \qquad p_0(s)=0, \qquad m=0,1,\ldots.. \tag{3.25}$$

Converge uniformly to the unique solution *p* of (1.2) on $[s_0,s_0+\rho]$, where $\rho=\min\{\alpha,\beta/N\}$, and N is the bound for h on T_0.

Proof: Since we proved uniqueness in theorem 3.1.1, it is enough to prove existence of solution by Arzela-Ascoli theorem.

Step:1 The consecutive approximations $\{p_{m+1}\}, m=0,1,2,\ldots$ given by (3.25) are well defined and continuous.

$$|p_{m+1}(s)|=\left|\int_{s_0}^{s} h(t,p_m(t))\Delta t\right| \le \int_{s_0}^{s}|h(t,p_m(t))|\Delta t \tag{3.26}$$

This gives the following result for

$$m=0,\ |p_1(s)| \le \int_{s_0}^{s}|h(t,p_0(t))|\Delta t \le Ns \le b \tag{3.27}$$

By induction, the sequence $\{p_{j+1}(s)\}$ is well defined and uniformly bounded on $[s_0,s_0+\rho]_S$.

Step: 2 To prove X is continuous function in $[s_0,s_0+\rho]_S$, where X is defined by

$$x(s)=\limsup_{j\to\infty}|p_j(s)-p_{j-1}(s)| \tag{3.28}$$

For

$$s_1,s_2 \in [s_0,s_0+\rho]_S,$$

we have

$$\|p_{j+1}(s_1)-p_j(s_1)\| \le \|p_{j+1}(s_2)-p_j(s_2)\|+2N|s_2-s_1| \tag{3.29}$$

Also

$$\begin{aligned}|p_{j+1}(s_1)-p_j(s_1)|-|p_{j+1}(s_2)-p_j(s_2)| &\le |p_{j+1}(s_1)-p_j(s_1)-p_{j+1}(s_2)+p_j(s_2)|\\ &\le \left|\int_{s_0}^{s_1}[h(t,p_j(t))-h(t,p_{j-1}(t))]\Delta t-\int_{s_0}^{s_2}[h(t,p_j(t))-h(t,p_{j-1}(t))]\Delta t\right|\\ &\le 2N\int_{s_1}^{s_2}\Delta t \le 2N(s_2-s_1)\end{aligned} \tag{3.30}$$

In (3.29), the right side expression in inequality is at most $X(s_2)+\varepsilon+2N(s_2-s_1)$ for large m if $\varepsilon>0$ provided that $|s_2-s_1|\le\frac{\varepsilon}{2N}.$

For some arbitrary ε and interchangeable s_1,s_2 we get

$$|X(s_1)-X(s_2)| \le 2N(s_2-s_1) \tag{3.31}$$

Hence X is continuous on $[s_0,s_0+\rho]_S$. By condition (A2) and definition of successive approximations, we get

$$|p_{j+1}(s)-p_j(s)| \le c\int_{s_0}^{s}|p_j(t)-p_{j-1}(t)|^{\alpha}\Delta t \tag{3.32}$$

The sequence $\{p_m\}$ is equicontinuous: that is $s_1,s_2\in[s_0,s_0+\rho]_S$ for each function p_m and some positive ε. If there exist $\gamma=\frac{\varepsilon}{N}$ such that $s_2-s_1\le\gamma,$ then

$$|p_{m+1}(s_1)-p_{m+1}(s_2)|=\left|\int_{s_1}^{s_2} h(t,p_m(t))\Delta t\right| \le \int_{s_1}^{s_2}|\bar{h}(t,p_m(t))|\Delta t \le N(s_1-s_2)\le\varepsilon \tag{3.33}$$

The family $\{p_j\}$ fulfills all conditions of Arzela Ascoli theorem in $c_{rd}[s_0,s_0+\rho]_S$. Hence there exists a subsequence $\{p_{jk}\}$ converging uniformly on $[s_0,s_0+\rho]_S$ as $j_k\to\infty.$ Let us assume

$$n^{*}(s)=\lim_{k\to\infty}|p_{jk}(s)-p_{jk-1}(s)| \tag{3.34}$$

If $\{\|p_j-p_{j-1}\|\}\to 0$ as $j\to\infty,$ then the limiting case of any subsequence is the only one solution [unique solution] p of (3.25). It follows that the entire sequence $\{p_j\}$ converges uniformly to $p.$

To show that $X\equiv 0.(ie)\ n^{*}(s)$ is null. Set

$$Q(s)=\int_{s_0}^{s}X(t)^{a}\,dt \tag{3.35}$$

and by denoting $Q^{*}(s)=s^{-k}X(s).$ To show that $\lim_{s\to 0^{+}}\phi^{*}(s)=0.$ Hence $\phi^{*}\equiv 0$ by absurdity.

Assume that $\phi^{*}(s)>0$ for $s\in]s_0,s_0+\rho]_S;$ then there exists s_1 such that

$$0<\bar{n}=\phi^{*}\left(s_1\right)=\max_{s\in[s_0,s_0+\rho]_S}\phi^{*}\left(s\right).$$

By condition (A1),

$$\bar{n}=\phi\left(s_1\right)=s_1^{-k}X\left(s_1\right)\le\bar{n}s_1<\bar{n} \qquad (3.36)$$

Which is contradiction. So $\phi^{*}=0.$ Hence (3.25) converge uniformly to a unique solution ϕ of (1.2) on $[s_0,s_0+\rho]_S$ by successive approximation.

3.3 Fractional order ODE and its uniqueness of Solution

Theorem 3.3.1. [Conditions of Krasnoselskii-Krein]

Denote $C_\nu([s_0,s_0+\alpha]_S,R)=\{p\mid p\in C[[s_0,s_0+\alpha]_S,R]\}$ and $(s-s_0)^{1-\nu}p\in C([s_0,s_0+\alpha]_S,R).$

Let $h(s,y)$ be continuous in T_0 and satisfying for all $(s,y),(s,\bar{y})\in T_0$

(C1) $\left|h(s,y)-h(s,\bar{y})\right|\le kl\,\Gamma(a)\left|s-s_0\right|^{-a}\left|y-\bar{y}\right|,\ s\ne s_0$

(C2) $\left|h(s,y)-h(s,\bar{y})\right|\le c\left|y-\bar{y}\right|^{\delta}$ where c,l,k are negative constants such that $k>1,\ kl\le a$ and $\frac{1}{k(1-\delta)}>1$, and all real numbers δ lies between 0 and 1. Then the fractional order initial value problem (1.3) has only one solution on $[s_0,s_0+\alpha]$.

Proof: Suppose *p* and *q* are two solutions of (1.3) in $[s_0,s_0+\alpha]_S.$ To show that $p\equiv q.$

To prove the result, define $\psi(s)$ and $Q(s)$ by

$$\left.\begin{aligned}&\psi(s)=\left|p(s)-q(s)\right|,\ \text{for every } s\in[s_0,s_0+\alpha]_S\\&Q(s)=\frac{c}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1}\tilde{\tilde{\psi}}^{\delta}(t)\,dt\ \text{ for every } s\in[s_0,s_0+\alpha]\end{aligned}\right\} \qquad (3.37)$$

Such that $\tilde{\tilde{\psi}}$ is the extension of ψ to the real interval $[s_0,s_0+\alpha].$ From condition (B2), it follows

$$\begin{aligned}\psi(s)&=\left|\frac{1}{\Gamma(a)}\int_{s_0}^{s}\left[h(t,p(t))-h(t,q(t))\right]\Delta t\right|\\&\le\frac{1}{\Gamma(a)}\int_{s_0}^{s}\left|h(t,p(t))-h(t,q(t))\right|\Delta t\le\frac{1}{\Gamma(a)}\int_{s_0}^{s}c(s-t)^{a-1}\left|\bar{p}(t)-\bar{q}(t)\right|^{\delta}dt=Q(s)\end{aligned} \qquad (3.38)$$

Also ${}_{s}^{S}D^{a}Q(s)=\hat{\psi}^{\delta}(s)=Q^{\delta}(s),$ for every $s\in[s_0,s_0+\alpha]_S.$ for every $s\in[s_0,s_0+\alpha]_S.$

By (3.37) and (3.38) and using lemma 2.6, we get for every

$$s\in[s_0,s_0+\alpha]_S,\ \psi(s)\le Q(s)=L(s-s_0)^{\xi} \qquad (3.39)$$

where L and ξ are defined in lemma 2.6.

Moreover, define

$$\phi(s)=\frac{\psi(s)}{(s-s_0)^{k}}.$$

We get

$$0\le\phi(s)\le L(s-s_0)^{\xi-ka}, \qquad (3.40)$$

for every

$$s\in[s_0,s_0+\alpha].$$

Hence

$$\lim_{s\to s_0}\phi(s)=0.$$

Therefore, if we define $\phi(s_0)=0,$ then the function is rd-continuous in $[s_0,s_0+\alpha]_S.$

Next to show that $\psi\equiv0.$ Assume contrarily ψ does not disappear at few points $S;$ that is $\psi(s)>0$ on $]s_0,s_0+\alpha]_S.$ Then there exists a maximum $n>0$ attained when S is equal to some $s_1:s_0<s_1\le s_0+\alpha$ such that $\phi(t)<n\le\phi(s_1),$ for $t\in[s_0,s_1)_S.$

By hypothesis (B1), we have

$$n=\phi\left(s_1\right)=\left(s-s_0\right)^{-k}\psi\left(s_1\right)$$

$$\begin{aligned}n&<(s-s_0)^{-k}\int_{s_0}^{s}kl(s-t)^{a-1}\left[h(t,p(t))-h(t,q(t))\right]\Delta t\\&\le(s_1-s_0)^{-k}\int_{s_0}^{s}kl(s-t)^{a-1}\frac{\psi(t)}{(t-s_0)^{-k}}\Delta t\\&\le(s_1-s_0)^{-k}\int_{s_0}^{s}kl(s-t)^{a-1}(t-s_0)^{k-a}\phi(t)\Delta t\le nkl(s_1-s_0)^{-a}\int_{s_0}^{s}(s_1-t)^{a-1}\Delta t\\&\le nkl(s_1-s_0)^{-a}\int_{s_0}^{s}(s_1-t)^{a-1}dt\le\frac{nkl}{a}<n\end{aligned} \qquad (3.41)$$

which is contradiction. Hence the solution is unique.

Theorem 3.3.2. Conditions of Kooi's

Let $h(s,y)$ be non-discontinuous in T_0 and for all $(s,y),(s,\bar{y})\in T_0$ satisfying

(D1) $\left|h(s,y)-h(s,\bar{y})\right|\le kl\Gamma(a)\left|s-s_0\right|^{-a}\left|y-\bar{y}\right|,\ s\ne s_0$

(D2) $\left|s-s_0\right|^{b}\left|h(s,y)-h(s,\bar{y})\right|\le c\left|y-\bar{y}\right|^{\delta}$ for some non-negative constants c,l and $k;$ also the non-imaginary positive numbers b,δ,k,l are such that $0<b<\delta<1$ and $k(1-\delta)<1-b$ & $kl\le a.$ Then, the first order initial value problem of first order FDE (1.3) has at most one solution on $[s_0,s_0+\alpha]_S.$

Proof: The proof of this theorem is similar to the last theorem 3.3.1.

3.4. Krasnoselskii-Krein Conditions on Time Lamina and Existence of Solution of FDE

Assume that (C1) and (C2) are satisfied; then the consecutive approximation towards solution is given by

$$p_{m+1}(s) = \int_{s_0}^{s} h(t, p_m(t))\Delta t \qquad p_0(s) = 0, \qquad m = 0,1,2,.... \tag{3.42}$$

tends to a finite limit uniformly to the unique solution p of (1.3) on $[s_0, s_0+\rho]$, where $\rho = \min\left\{\alpha, \left(\frac{\beta\Gamma(1+a)}{N}\right)^{\frac{1}{a}}\right\}$ and N is the bound for h on T_0. (3.43)

Proof: Since uniqueness of the solution have been proved by theorem 3.3.1, we have to prove the existence of solution by Arzela Ascoli theorem. The successive approximation $\{p_{m+1}\}, m = 0,1,2,...$ given in (3.42) are properly defined and continuous.

$$|p_{m+1}(t)| = \left|\frac{1}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1} h(t, p_m(t))\Delta t\right| \le \frac{1}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1}|h(t, p_m(t))|\Delta t \tag{3.44}$$

$$\text{For } n = 0, \; |p_1(s)| \le \frac{N}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1}\Delta t \le \frac{N}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1}dt \le \frac{N\alpha^a}{\Gamma(a+1)} \le \beta \tag{3.45}$$

By mathematical induction, the flow of sequence $\{p_{j+1}(s)\}$ is properly defined and uniformly bounded on $[s_0, s_0+\rho]_S$.

Step: 2 To prove *X* is continuous function in $[s_0, s_0+\rho]_S$, where *X* is defined by

$$X(s) = \limsup_{j\to\infty} |p_j(s) - p_{j-1}(s)| \tag{3.46}$$

For $s_1, s_2 \in [s_0, s_0+\rho]_S$, we have

$$|p_{j+1}(s_1) - p_j(s_1)| \le |p_{j+1}(s_2) - p_j(s_2)| + \frac{4N}{\Gamma(a+1)}(s_2 - s_1)^a \tag{3.47}$$

That is

$$\begin{aligned}
&|p_{j+1}(s_1) - p_j(s_1)| - |p_{j+1}(s_2) - p_j(s_2)| \le |p_{j+1}(s_1) - p_j(s_1) - p_{j+1}(s_2) + p_j(s_2)| \\
&\le \frac{1}{\Gamma(a)}\left|\int_{s_0}^{s_1}(s_1-t)^{a-1}\left[h(t,p_j(t)) - h(t,p_{j-1}(t))\right]\Delta t - \right. \\
&\qquad \left. - \int_{0}^{s_2}(s_2-t)^{a-1}\left[h(t,p_j(t)) - h(t,p_{j-1}(t))\right]\Delta t\right| \\
&\qquad - \int_{0}^{s_2}(s_2-t)^{a-1}\left[h(t,p_j(t)) - h(t,p_{j-1}(t))\right]\Delta t\Bigg| \\
&\le \frac{2N}{\Gamma(a)}\left[\left|\int_{0}^{s_1}\left((s_1-t)^{a-1} - (s_2-t)^{a-1}\right)\Delta t - \int_{s_1}^{s_2}(s_2-t)^{a-1}\Delta t\right|\right] \\
&\le \frac{2N}{\Gamma(a)}\left[\left|\int_{0}^{s_1}\left((s_1-t)^{a-1} - (s_2-t)^{a-1}\right)dt - \int_{s_1}^{s_2}(s_2-t)^{a-1}dt\right|\right] \qquad (3.48) \\
&\le \frac{2N}{a\Gamma(a)}\left[s_1^a - s_2^a + 2(s_2-s_1)^a\right] \le \frac{4N}{\Gamma(a+1)}(s_2-s_1)^a
\end{aligned}$$

The right side of inequality (3.47) is at most $|\cdot| - \varepsilon + \left[\frac{4N}{\Gamma(a+1)}|s_2 - s_1|\right]$ for large m if $\varepsilon > 0$ given that $|s_2 - s_1| \le \left[\frac{\varepsilon\Gamma(a+1)}{4N}\right]^{\frac{1}{a}}$.

Since ε is arbitrary and s_1, s_2 can be interchangeable, then

$$|X(s_1) - X(s_2)| \le \frac{4N}{\Gamma(a+1)}(s_2 - s_1). \tag{3.49}$$

That is *X* continuous on $[s_0, s_0+\rho]_S$.

By condition (C2) and the definition consecutive

approximations, we get

$$|p_{j+1}(s) - p_j(s)| \le \frac{c}{\Gamma(a)}\int_{s_0}^{s}\left[|p_j(t) - p_{j-1}(t)|^a\right]\Delta t \tag{3.50}$$

therefore the sequence $\{p_m\}$ is equicontinuous. For each function p_m and $\varepsilon > 0$, $s_1, s_2 \in [s_0, s_0+\rho]_S$. If there exists $\gamma = \frac{\varepsilon^{-a}\Gamma(a+1)}{N} \ni s_2 - s_1 \le \gamma;$ then $|p_{m+1}(s_1) - p_{m+1}(s_2)| \le \frac{2N}{\Gamma(a+1)}(s_1 - s_2)^a \le \varepsilon$.

Let us denote $n^*(s) = \lim_{k\to\infty}|p_{jk}(s) - p_{jk-1}(s)|$. Further, if $\{|p_j - p_{j-1}|\} \to 0 \text{ as } j \to \infty$, then the limiting case of any subsequence is the unique solution *p* of (3.42).

Let $Q(s) = \frac{c}{\Gamma(a)}\int_{s_0}^{s}(s-t)^{a-1}X(t)^a dt$ and define $\phi^*(s) = s^{-k}X(s)$ and then using lemma 2.6, we get that $\phi(s) \le L(s_1 - s_0)^{j-ka}$ which gives that $\lim_{s\to 0^+}\phi^*(s) = 0$. And also proved that $\phi^* \equiv 0$ by absurdity. presume that $\phi^*(s) > 0$ at any point in $[s_0, s_0+\rho]_S$; then there exist s_1 such that $0 < \bar{n} = \phi^*(s_1) = \max_{s_1\in[s_0, s_0+\rho]_S}\phi^*(s)$. For condition (C1), we obtain

$$n = \phi(s_1) = (s_1 - s_0)^{-ka}\psi(s_1) \le (s_1 - s_0)^{-ka}\int_{s_0}^{s_1} kl(s_1 - t)^{a-1}(t - s_0)^{-a}\psi(t)dt$$

$$n \le kl(s_1 - s_0)^{-ka}\int_{s_0}^{s_1}(s_1 - t)^{a-1}(t - s_0)^{ka-a}\phi(t)dt < kl\,n(s_1 - s_0)^{-a}\int_{s_0}^{s_1}(s_1 - s_0)^{a-1}dt < \frac{kl\,n}{a} < n.$$

this is an inconsistency. (i.e.) $\phi^* = 0.$ Hence Picard's successive approximation (3.42) tends to finite limit (uniform convergence) to unique solution *p* of (1.2) on $[s_0, s_0+\rho]_S$.

CONCLUSION

Hence, we can establish the solution of non-linear FDE with order $a \in (0,1]$ by few basic named conditions.

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