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A Study on the First Eigenvalue of the P-Laplacian and Critical Sets of Harmonic Functions by Defining Geometric P-Laplacian on Riemannian Manifolds

Manisha Rajput*

Research Scholar, Mansarovar Global University, Sehore (MP)

Abstract – In mathematics, the p-Laplacian, or the p-Laplace administrator, is a quasilinear elliptic incomplete differential administrator of second request. It is a nonlinear speculation of the Laplace administrator, where p is permitted to go over 1<p<∞. The properties of the spectrum of the weighted p-Laplacian on a complete Riemannian complex with evolving geometry it is a notable component that spectrum as an invariant amount advances as the space does under any geometric flow. The variation recipes, monotonicity, and differentiability for the first eigen value of the p-Laplacian on a n-dimensional shut Riemannian complex whose measurement develops by a summed up geometric flow. the spectrum of the Laplacian on noncompact non-complete manifolds also attracts attention of mathematicians and physicists in the past three decades, since the investigation of the spectral properties of the Dirichlet Laplacian in infinitely extended regions has applications in elasticity and so on. The PDEs involving p-Laplacian are considered in differential geometry in the investigation of critical points for p-harmonic maps between Riemannian manifolds and the eigenvalue problems for p-Laplacian on Riemannian manifolds serve for estimations of the diameter of the manifolds. By using the theory of self-adjoint operators, the spectral properties of the linear Laplacian on a domain in a Euclidean space or a manifold have been concentrated broadly.

Keywords – Eigen Value, P-Laplacian, Harmonic Functions, Geometric, Riemannian, Manifolds, etc.

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1. INTRODUCTION

In science, the *p*-Laplacian, or the *p*-Laplace operator, is a quasilinear elliptic partial differential operator of second request. It is a nonlinear generalization of the Laplace operator, where p is permitted to range over $1<p<\infty$. In science, numerical physics and the hypothesis of stochastic processes, a harmonic function is a twice continuously differentiable function $f: U \to R$, where U is an open subset of R^n, which fulfills Laplace's equation. The properties of the spectrum of the weighted *p*-Laplacian on a complete Riemannian complex with evolving geometry it is a notable component that spectrum as an invariant amount advances as the space does under any geometric flow. All through, we will consider a *n*-dimensional complete Riemannian complex $(M, g, d\mu)$ equipped with weighted measure $d\mu=e^{-\phi}dv$ and potential function $\phi \in C\infty(M, d\mu)$, whose metric $g=g(t)$ advances along

Either the Ricci-harmonic flow or volume preserving Ricci-harmonic flow

2. THE FIRST EIGENVALUE OF *p*-LAPLACIAN ON EVOLVING GEOMETRY AND APPLICATIONS

The variation method, monotonicity, and differentiability for the first eigen value of the *p*-Laplacian on a *n*-dimensional shut Riemannian complex whose measurement develops by a summed up geometric flow. It is demonstrated that the first nonzero eigen value is monotonically non-diminishing along the flow under certain geometric conditions and that it is differentiable all over the place. These outcomes provide a brought together approach to the investigation of eigen value variations and applications under numerous geometric flows.

2.1 Geometric flow

Let (M, g) be an *n*-dimensional shut Riemannian complex ($n>1$). Let $g(x, t)$ be a one parameter group of measurements for $t \in [0, T]$ and $x \in M$. We state that $g(x, t)$ is a summed up geometric flow in the event that it advances by the accompanying equation with $g(x,0) = g_0(x)$, where $0<T<T_\varepsilon$ is the maximal time of presence, i.e., T_ε is the first time where the flow explodes and h is an overall time-dependent symmetric 2-tensor. Here h is thought to be smooth in the two factors t and x. This is clear since g is likewise smooth in the two factors. The scaling factor 2 in (1) is immaterial while the negative sign might be important in some specific applications to keep the flow either forward or in reverse in time.

$$\frac{\partial}{\partial t}g(x,t) = -2h(x,t), \quad (x,t) \in M \times [0,T] \qquad (1)$$

Two popular examples of geometric flows in this class are: the Ricci flow with h being the Ricci shape tensor, and the mean bend flow with $h = H\Pi$ (where H is the mean arch and Π is the second crucial structure on M). Different examples incorporate Yamabe flow, Ricci-harmonic flow, RicciBourguignon flow. One may impose boundedness condition on tensor h. Truth be told, such boundedness and sign assumptions on h are preserved as long as the flow exists, so it follows that the measurements are consistently same. Precisely, if $-K_1 g \le h \le K_2 g$, where $g(t)$, $t \in [0, T]$ is the flow, at that point

$$e^{-K_1 T} g(0) \le g(t) \le e^{K_2 T} g(0).$$

To see the last limits, we consider the evolution of a vector structure $|X|_g = g(X, X)$, $X \in T_x M$. From (1) and the boundedness of the tensor h, we have $|\partial_t g(X, X)| \le K_2 g(X, \boldsymbol{X})$, which implies (by coordinating from t_1 to t_2)

$$\left|\log \frac{g(t_2)(X,X)}{g(t_1)(X,X)}\right| \le K_2 t\Big|_{t_1}^{t_2}.$$

Taking the exponential of this gauge with t_1 =0 and t_2 =T yields $|g(t)| \le e^{K_2 T} g(0)$ from which the uniform boundedness of the measurement follows. In this manner, if there holds boundedness assumption

$$-K_1 g \le h \le K_2 g$$

The metric $g(t)$ are consistently limited underneath and above forever $0 \le t \le T$ under the geometric flow (1). At that point, it doesn't make a difference what metric we use in the contention that follows.

2.2 Eigen value of *p*-Laplacian

The p-Laplace operator is characterized by

$$\Delta_{p,g} f(x) := \mathrm{div}(|\nabla f|^{p-2} \nabla f)(x)$$

For $p \in [1, \infty)$, where div is the difference operator, and the adjoint of inclination (graduate) for the L^2 -standard instigated by g on the space of differential structures at the point when p =2, $\Delta_{2,g}$ is the standard Laplace-Beltrami operator. The eigen values and the corresponding eigen functions of $\Delta_{p,g}$ fulfill the accompanying nonlinear eigen value problem

$$\Delta_{p,g} f = -\lambda |f|^{p-2} f, \quad f \ne 0. \quad (2)$$

It is notable that the principal image of (2) is nonnegative all over and carefully positive at the neighborhood of the point where $\nabla f \ne 0$. We likewise realize that (2) has feeble solutions with only partial routineness of class $C^{1,\alpha}$,$(0 < \alpha < 1)$ as a rule. Intrigued perusers can locate the traditional papers by Evans and Tolksdorff. Notice that the least eigen value of $\Delta p,g$ on shut complex is zero with the corresponding eigen function being a constant. Subsequently, we allude to the infimum of the positive eigen values as the first nonzero eigen value or simply the first eigen value. The first eigen value of $\Delta p, g$ is portrayed by the min-max principle

$$\lambda_{p,1} = \inf_f \left\{ \frac{\int_M |\nabla f|_g^p \, d\mu_g}{\int_M |f|_g^p \, d\mu_g} \;\middle|\; f \ne 0,\; f \in W^{1,p}(M) \right\} \qquad (3)$$

Satisfying the following constraint $\int_M |f|_g^{p-2} f d\mu_g = 0$, where $d\mu_g$ is the volume measure on (M,g). Clearly, the infimum doesn't change when one replaces $W^{1,p}(M)$ by $C^\infty(M)$. The corresponding Eigen function is the energy minimizer of Rayleigh remainder (3) and fulfills the accompanying Euler-Lagrange equation

$$\int_M [|\nabla f|^{p-2} \langle \nabla f, \nabla \phi \rangle - \lambda |f|^{p-2} \langle f, \phi \rangle] d\mu_g = 0 \qquad (4)$$

For $\varphi \in C_0^\infty(M)$ in the cognition of distribution it is notable that p-Laplacian has discrete eigen values yet remains obscure whether it only has discrete eigen values for limited connected domains. Other notable outcomes reveal to us that the first nonzero eigen value is simple and confined. Here the simplicity shows that any nontrivial eigen function corresponding to $\lambda_{p,1}$ doesn't change sign and that any two first eigen functions are constant multiple of one another. In contrast to the spectrum of the Laplace-Beltrami operator (the case p=2), the p-Laplacian is nonlinear as a rule. Besides, it isn't known whether $\lambda_{p,1}$ or its corresponding eigen function is C^1-differentiable (or even locally Lipschitz) along any geometric flow of the structure (1). Notwithstanding, it has been pointed out that the differentiability for the case p=2 is a consequence of eigen value perturbation theory; see, for instance, and the references therein. Consequently, any approach that accepts differentiability of eigen values and eigen functions under the flow must be applied to the case p=2.

Presently to keep away from the differentiability assumption on the first eigen value and the corresponding eigen function for the situation $p \ne 2$, we will apply strategies introduced in under the Ricci flow to examine the variation and monotonicity of $\lambda_{p,1}(t) = \lambda_{p,1}(t, f(t))$, where $\lambda_{p,1}(t, f(t))$ and $f(t)$ are thought to be smooth. The evolution and the monotonicity formulas for the first eigen value inferred here don't depend on the evolution of the eigen function. The eigen function only requirements to fulfill certain normalization condition. There are numerous outcomes on the evolution and monotonicity of Eigen values of the Laplace operator on evolving manifolds with or without ebb and flow assumptions. One can find under the Ricci flow, under Ricci-harmonic flow and along unique geometric flow with entropy techniques. The investigation of the properties of eigen values of the *p*-Laplacian on evolving complex is still youthful. The main point of this paper is to investigate if those known properties of $\lambda_{p,1}$ on static measurement and for the case *p*=2 on evolving metric can be stretched out to different geometric flows. We anyway intend to develop a brought together calculation that can be utilized for this purpose on time-dependent measurements. Many interesting outcomes concerning the conduct of $\lambda_{p,1}$ can be found in for static measurements and for evolving measurements along different geometric flows.

3. GEOMETRIC PROPERTIES OF SOLUTION TO THE ANISOTROPIC *p*-LAPLACE EQUATION IN DIMENSION TWO

We consider solutions to the equation (5) with a consistently elliptic and Lipschitz continuous symmetric lattice, in dimension two. We consider solutions $u \in W^{1,p}_{loc}(\Omega)$ to the following savage elliptic equation which we will call the anisotropic *p*-Laplace equation

$$\operatorname{div}(|A\nabla u \cdot \nabla u|^{(p-2)/2} A\nabla u) = 0 \text{ in } \Omega \quad (5)$$

Where Ω is a two-dimensional domain, p fulfills $1<p<\infty$, and $A = A(x)$ is a symmetric network satisfying hypotheses of uniform ellipticity and of Lipschitz continuity. Equation (5) can be seen as the Euler equation for the variational integral

$$J(u) = \int_\Omega |A\nabla u \cdot \nabla u|^{p/2}\, dx \qquad (6)$$

And its interest arises from various applied contexts related to composite materials, (for example, nonlinear dielectric composites, whose nonlinear behavior is displayed by the supposed power-law. In such a case many things are thought about the local behavior of solutions and about the structure of level lines and critical points. First, the Hartman and Wintner theorem [HW] discloses to us that for each $x^0 \in \Omega$, and up to a linear change of coordinates which renders $A(x^0)$= const. I, $u(x) - u(x^0)$ is asymptotic to a homogeneous harmonic polynomial of $x-x^0$, and this asymptotics carries over to first request derivatives. From this basic fact, one can infer that in the event that *u* is non-identically constant, then its critical points are isolated. Besides, on the off chance that x^0 is a zero of multiplicity m for ∇u, then the level set $\{x \mid u(x) = u(x^0)\}$ is composed, near x^0, by exactly *m*+1 simple arcs intersecting at x^0 only. Next, it is possible to evaluate the number, and the multiplicities, of critical points of a solution as far as properties of its Dirichlet data [A1], [A2], or of other types of boundary data [AM1]. Such outcomes have also been generalized to weak solutions *u* to (6) when the coefficient matrix A is simply limited measurable [AM2].

4. EIGENVALUE INEQUALITIES FOR THE *P*-LAPLACIAN ON A RIEMANNIAN MANIFOLD AND ESTIMATES FOR THE HEAT KERNEL

By using the theory of self-adjoint operators, the spectral properties of the linear Laplacian on a domain in a Euclidean space or a manifold have been concentrated widely. Mathematicians generally are interested in the spectrum of the Laplacian on compact manifolds (with or without boundary) or non-compact complete manifolds, since in these two cases the linear Laplacians can be particularly reached out to self-adjoint operators. Nonetheless, the spectrum of the Laplacian on noncompact non-complete manifolds also attracts attention of mathematicians and physicists in the past three decades, since the investigation of the spectral properties of the Dirichlet Laplacian in infinitely extended regions has applications in elasticity, acoustics, electromagnetism, quantum physics, and so forth As of late, the author has proved the presence of discrete spectrum of the linear Laplacian on a class of 4-dimensional rotationally symmetric quantum layers, which is non-compact non-complete manifolds, in under some geometric assumptions therein.

A natural generalization of the linear Laplacian is the purported *p*-Laplacian underneath. Although many outcomes about the linear Laplacian (*p*=2) have been obtained, many rather basic questions about the spectrum of the nonlinear *p*-Laplacian remain to be addressed. Let Ω be a limited domain on a n-dimensional Riemannian manifold (*M, g*). We consider the following nonlinear Dirichlet eigen value problem.

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in}\,\Omega, \\ u = 0 & \text{on}\,\partial\Omega, \end{cases}$$

Where

$$\Delta_p u = \operatorname{div}(|\nabla u|_g^{p-2} \nabla_u)$$

is the *p*-Laplacian with $1<p<\infty$. In local coordinates $\{x_1, \ldots, x_n\}$ on *M*, we have

$$\Delta_p u = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det(g_{ij})}\, g^{ij} |\nabla u|^{p-2} \frac{\partial u}{\partial x_j}\right)$$

Where

$$|\nabla u|^2 = |\nabla u|_g^2 = \sum_{i,j=1}^{n} g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}$$

And $(g^{ij})=(g^{ij})^{-1}$ is the inverse of the metric matrix.

5. MAXIMUM RELATING RADIAL *p*-LAPLACIAN BY APPLICATIONS TO NONLINEAR EIGEN VALUE PROBLEMS

Problems involving p-Laplace operator are subject of intensive investigations as they illustrate many of phenomena that happen in nonlinear analysis. Among their applications are singular and non-singular boundary value problems which appear in various branches of mathematical physics. They arise as a model example in the liquid dynamics; glaciology; stellar dynamics; in the theory of electrostatic fields; in the more general context in quantum physics; in the nonlinear elasticity theory as a basic model; and many others. The PDEs involving *p*-Laplacian are considered in differential geometry in the investigation of critical points for p-harmonic maps between Riemannian manifolds and the eigenvalue problems for *p*-Laplacian on Riemannian manifolds serve for estimations of the diameter of the manifolds. Eigenvalue problems involving *p*-Laplacian are applied in functional analysis to infer sharp Poincar'e and Writinger type inequalities, Sobolev embeddings and isoperimetric inequalities. Geometric properties of p-harmonic functions play significant part in the theory of Carnot–Caratheodory groups like Heisenberg group and in the analysis on measurement spaces. One of the problems we experience when investigating p-harmonic equation is that not many explicit solutions are known – affine, quasiradial, radial. Among them radial solutions form the most stretched out nontrivial class in which many properties of p-harmonic world can be recognized? Another motivation to examine radial solutions comes from the seminal paper by Gidas, Ni and Nirenberg who expanded Serrin's moving plane strategy from and proved that at times only radial solutions are admitted. Additionally, it can happen that among the solutions of the PDE are the radial ones regardless of whether the radial solutions are by all account not the only ones. We shall consider radial solutions of the equation

$$-a(|x|)\mathrm{div}(|\nabla w(x)|^{p-2}\nabla w(x)) = \varphi(w(x)) \text{ a.e. in } B = B(0,R) \subset R^n \qquad (7)$$

We assume that $p>1$, $n>1$, $R\in(0,\infty]$ (for $R=\infty$ the above equation is defined on R^n), $a(.)$ is nonnegative and belongs to a certain class of functions which will be depicted later, while φ is an arbitrary odd continuous function with the end goal that τφ(τ) is of constant sign for L^1 almost all τ '*s*. In general our equation is given in a non-divergent form.

6. CONCLUSION

In mathematics, the *p*-Laplacian, or the *p*-Laplace operator, is a quasilinear elliptic partial differential operator of second request. It is a nonlinear generalization of the Laplace operator, where p is allowed to range over $1<p<\infty$. The variation formulas, monotonicity and differentiability for the first eigenvalue of the *p*-Laplacian on an n-dimensional shut Riemannian manifold whose measurement develops by a generalized geometric flow. By using the theory of self-adjoint operators, the spectral properties of the linear Laplacian on a domain in a Euclidean space or a manifold have been concentrated broadly. Mathematicians generally are interested in the spectrum of the Laplacian on compact manifolds (with or without boundary) or non-compact complete manifolds, since in these two cases the linear Laplacians can be extraordinarily stretched out to self-adjoint operators. Problems involving *p*-Laplace operator are subject of intensive investigations as they illustrate many of phenomena that happen in nonlinear analysis. Among their applications are singular and nonsingular boundary value problems which appear in various branches of mathematical physics.

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Corresponding Author

Manisha Rajput*

Research Scholar, Mansarovar Global University, Sehore (MP)