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# Fixed Point Theory in Metric Spaces

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***Abstract - Using fixed point theorems, the primary objective of this study is to provide existence results and stability conditions for a class of fractional order differential equations. Existence findings are derived from Schauder's fixed point theorem and the Banach contraction principle. In addition, the use of Krasnoselskii's fixed point theorem to develop stability conditions for a particular class of fractional order differential equations is given a lot of attention. The usefulness of the stability result is shown via the use of an example. Through using the characteristics of -distance mappings and -admissible mappings, we present the idea of generalized contraction mappings and show the existence of a fixed-point theorem for such mappings. This is accomplished by mapping properties. In addition, we extend our conclusion to the theorems of coincidence point and common fixed point in metric spaces. Further, the fixed-point theorems that are endowed with an arbitrary binary relation may also be deduced from our conclusions thanks to this line of reasoning.***

***Keywords - fixed, theorems***

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## INTRODUCTION

It is common knowledge that many problems in various subfields of mathematics may be converted into a fixed point problem of the form Tx = x for self-mapping T defined on the framework of metric space (X, d). This transformation can be done in a variety of ways. Banach presented the idea of contraction mapping in 1992 and later proved the fixed point theorem for such mapping, which is now known as the Banach contraction principle. These contributions paved the way for more research and development in the area of analysis. A number of mathematical experts made use of a variety of conditions on self-mappings in order to demonstrate a number of fixed point theorems in metric spaces and other spaces.

1969 saw the establishment of the fixed point theorem for multivalued contraction mapping by Nadler. This was accomplished by the use of the idea of Hausdorff metric, which is an extension of the traditional Banach contraction principle. After that, Kaneko generalized the findings of Jungck by extending the analogous findings of Nadler to include both single valued mapping and multivalued mapping. Following this, there are a number of findings that expand the scope of this conclusion in a variety of new ways.

On the other hand, Kada and colleagues presented the idea of w-distance in relation to a metric space. They were able to enhance Caristi's fixed point theorem, Ekland's variational principle, and Takahashi's existence theorem by using this approach. After that, Suzuki and Takahashi came up with the fixed-point solution for the multivalued mapping with regard to the w-distance. In point of fact, this conclusion is an enhancement of the fixed-point theorem proposed by Nadler. In the context of metric spaces, several mathematicians have used w distance to establish a number of fixed-point theorems; for example, see. Recently, Kutbi produced a helpful lemma for w-distance, which is an improved version of the lemma provided in, and demonstrated a crucial lemma on the presence of f-orbit for extended f-contraction mappings. Both of these were accomplished by proving a key lemma on the existence of f-orbit. In addition to this, he demonstrated the existence of coincidence points as well as common fixed points for generalized f-contraction mappings that did not include the extended Hausdorff metric.

The objective of this study is to present the generalized w-contraction mapping and to demonstrate the fixed point theorem for such a mapping by utilizing the concept of -admissible mapping proposed by Mohammadi et al. This concept is a multivalued mapping version of -admissible mapping proposed by Samet et al. and is distinct from the concept of -admissible that has been presented in. Our findings also have ramifications for the applications of coincidence point and common fixed-point theorems in metric spaces, as well as fixed point theorems endowed with an arbitrary binary relation. Our findings enhance and supplement the primary conclusion reached by Kutbi as well as several findings found in the aforementioned body of research.

## OBJECTIVES OF THE STUDY

1. To study on fixed point theorems

2. To study on g-metric space by using clrg property

## FIXED POINT THEOREMS

Fixed point theorems concern maps $f$ of a set $X$ into

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, x_{n+m-1}) + \cdots + d(x_{n+1}, x_n)$$
$$\leq (\lambda^{n+m} + \cdots + \lambda^n) d(f(x_0), x_0)$$
$$\leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0).$$

itself that, under certain conditions, admit a *fixed point*, that is, a point $x$ $X$ in such a way that f (x) = x. The knowledge that fixed points do in fact exist has important applications in a wide variety of subfields within analysis and topology. Let us demonstrate this by way of the following, which is a simple yet representative example.

Example **Let us assume that we have been provided with a set of n equations in n unknowns that are of the type.**

$g_j(x_1, \ldots, x_n) = 0, j = 1, \ldots, n$

where the $g_j$ are continuous real-valued functions of the real variables $x_j$. Let $h_j(x_1, \ldots, x_n) = g_j(x_1, \ldots, x_n) + x_j$, and for any point $x = (x_1, \ldots, x_n)$ define $h(x) = (h_1(x), \ldots, h_n(x))$. Assume now that h *has* a fixed-point $\bar{x} \in R^n$. Then it is easily seen that $\bar{x}$ is a solution to the system of equations.

**In the next chapter, a number of different applications of fixed-point theorems will be discussed.**

## THE BANACH CONTRACTION PRINCIPLE

**Definition** Let $X$ be a metric space equipped with a distance $d$. A map $f : X \to X$ is said to be

*Lipschitz continuous* if there is $\lambda \geq 0$ such that

$d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2), \quad \forall x_1, x_2 \in X.$

The smallest $\lambda$ for which the above inequality holds is the *Lipschitz constant* of $f$. If $\lambda \leq 1$ $f$ is said to be *non-expansive*, if $\lambda < 1$ $f$ is said to be a *contraction*.

**Theorem[Banach]** *Let f be a contraction on a complete metric space Then f has a uniquefixed point* $\bar{x} \in X$.

PROOF Notice first that if $x_1, x_2 \in X$ are fixed points of $f$, then

$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \lambda d(x_1, x_2)$

**which imply $x_1 = x_2$. Choose now any $x_0 \in X$, and define the iterate sequence**

$x_{n+1} = f(x_n)$. By induction on $n$, $d(x_{n+1}, x_n) \leq \lambda^n d(f(x_0), x_0)$.

If $n \in N$ and $m \geq 1$,

**Hence $x_n$ is a Cauchy sequence, and admits a limit $\bar{x} \in X$, for $X$ is complete. Since $f$ is**

continuous, we have $f(\bar{x}) = \lim_n f(x_n) = \lim_n x_{n+1} = \bar{x}$.

Remark **Notice that letting $m \to \infty$ in (1) we find the relation**

$$d(x_n, \bar{x}) \leq \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0)$$

Which provides a control on the convergence rate of $x_n$ to the fixed point $\bar{x}$. The completeness of X is quite important to this scenario. In point of fact, contractions performed on incomplete metric spaces may not have any fixed points.

**Example** Let $X = (0, 1]$ with the usual distance. Define $f : X$ $X$ as

$f(x) = x/2$.

**Corollary** *Let X be a complete metric space and Y be a topological space. Let f : X Y X be acontinuous function. Assume that f is a contraction on X uniformly in Y , that is,*

$d(f(x_1, y), f(x_2, y)) \leq \lambda d(x_1, x_2), \quad \forall x_1, x_2 \in X, \forall y \in Y$

*for some $\lambda < 1$. Then, for every fixed $y \in Y$, the map $x \mapsto f(x, y)$ has a uniquefixed point $\phi(y)$.Moreover, the function $y \mapsto \phi(y)$ is continuous from $Y$ to $X$.*

Notice that if $f : X \times Y \to X$ is continuous on $Y$ and is a contraction on $X$ uniformly in $Y$ ,then $f$ is in fact continuous on $X \times Y$ .

PROOF In light of Theorem 1.3, we only have to prove the continuity of $\phi$. For

$$\begin{aligned} d(\varphi(y),\varphi(y_0)) &= d(f(\varphi(y),y),f(\varphi(y_0),y_0)) \\ &\leq d(f(\varphi(y),y),f(\varphi(y_0),y)) + d(f(\varphi(y_0),y),f(\varphi(y_0),y_0)) \\ &\leq \lambda d(\varphi(y),\varphi(y_0)) + d(f(\varphi(y_0),y),f(\varphi(y_0),y_0)) \end{aligned}$$

$y, y_0 \in Y$ , we have

**Which implies?**

$$d(\varphi(y),\varphi(y_0)) \leq \frac{1}{1-\lambda} d(f(\varphi(y_0),y),f(\varphi(y_0),y_0)).$$

Since the above right-hand side goes to zero as $y \to y_0$, we have the desired continuity.

Remark If in addition $Y$ is a metric space and $f$ is Lipschitz continuous in $Y$ , uniformly with respect to $X$, with Lipschitz constant $L \geq 0$, then the function $y \mapsto \phi(y)$ is Lipschitz continuous with the Lipschitz constant being equal to or less than $L/(1-\lambda)$.

The necessary condition that must be met by f in order to have a single fixed point is presented in Theorem.

**Example** Consider the map

$$g(x) = \begin{cases} 1/2 + 2x & x \in [0, 1/4] \\ 1/2 & x \in (1/4, 1] \end{cases}$$

Mapping onto itself. Then $g$ is not even continuous, but it has a unique fixed point ($x = 1/2$).

The following corollary takes into consideration the circumstances described above and demonstrates the existence of fixed points as well as their singularity under more broad conditions. Definition For $f : X \to X$ and $n \in \mathbb{N}$, we denote by $f^n$ the $n^{th}$ *-iterate* of $f$ , namely, $f \circ \cdots \circ f$ $n$-times ($f^0$is the identity map).

**Corollary** *Let $X$ be a complete metric space and let $f : X \to X$. If $f^n$ is a contraction, for some $n \geq 1$, then* $f$ *has a unique fixed point $\bar{x} \in X$.*

PROOF Let $\bar{x}$ be the unique fixed point of $f^n$, given by Theorem 1.3. Then

$f^n(f(\bar{x})) = f(f^n(\bar{x})) = f(\bar{x})$, which implies $f(\bar{x}) = \bar{x}$. Since a fixed point of $f$ is clearly a fixed point of $f^n$, we have uniqueness as well

**Notice that in the example $g^2(x) \equiv 1/2$.**

**Additional examples of the contraction principle extensions** There are a significant number of different extensions of Theorem that may be found in the academic literature. In this section, we will highlight certain findings.

**Theorem [Boyd-Wong] Let** *$X$ be a complete metric space, and let f: $X$ $X$. Assume there*

*exists a right-continuous function $\phi$: [0, ) [0, ) such that $\phi(r) < r$ if $r > 0$, and*$\rightarrow$

$d(f(x_1), f(x_2)) \leq \phi(d(x_1, x_2))$, $\forall x_1, x_2 \in X$.

$\infty$ $\rightarrow$ $\infty$

*Then f has a unique fixed point $\bar{x}$ $X$. Moreover, for any $x_0$ $X$ the sequence*

$f^n(x_0)$ *converges to* $\bar{x}$. $\in$

**Clearly, Theorem is a particular case of this result, for $\phi(r) = \lambda r$.**

PROOF IF $x_1, x_2 \in X$ are fixed points of $f$, then

$$d(x_1, x_2) = d(f(x_1), f(x_2)) \leq \phi(d(x_1, x_2))$$

**so $x_1 = x_2$. To prove the existence, fix any $x_0 \in X$, and define the iterate sequence $x_{n+1} = f(x_n)$. We show that $x_n$ is a Cauchy sequence, and the desired conclusion follows arguing like in the proof of Theorem. For $n$ 1, define the positive sequence**

$$a_n = d(x_n, x_{n-1}).$$

**It is clear that $a_{n+1} \leq \phi(a_n) \leq a_n$; therefore, $a_n$ converges monotonically to some $a \geq 0$. From the right- continuity of $\phi$, we get $a \leq \phi(a)$, which entails $a = 0$. If$x_n$ is not a Cauchy sequence, there is $\varepsilon > 0$ and integers $m_k > n_k \geq k$ for every ≥1 such that**

$$d_k := d(x_{m_k}, x_{n_k}) \geq \varepsilon, \qquad \forall k \geq 1.$$

In addition, upon choosing the smallest possible $m_k$, we may assume that

$d(x_{m_k-1}, x_{n_k}) < \varepsilon$

For $k$ big enough (here we use the fact that $a_n \to 0$). Therefore, for $k$ big enough,

$\varepsilon \le d_k \le d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) < a_{m_k} + \varepsilon$

Implying that $d_k \to \varepsilon$ from above as $k \to \infty$ Moreover,

$d_k \le d_{k+1} + a_{m_{k+1}} + a_{n_{k+1}} \le \phi(d_k) + a_{m_{k+1}} + a_{n_{k+1}}$

and taking the limit as $k \to \infty$ we obtain the relation $\varepsilon \le \phi(\varepsilon)$, which has to be false since $\varepsilon > 0$.

**Theorem [Caristi]** *Let there be a full metric space denoted by X, and let there also be. $f : X \to X$. Take it for granted that there is a lower semicontinuous function $\psi$: $X \to [0, \infty)$ such that*

$d(x, f(x)) \le \psi(x) - \psi(f(x)), \quad \forall x \in X.$

*Then f has (at least) a fixed point in X.*

**Again, Theorem 1.3 is a particular case, obtained for $\psi(x) = d(x, f(x))/(1-\lambda)$.**

**Notice that *f* need not be continuous.**

PROOF WE PROVIDE AN INCOMPLETE ORDERING ON X, USING THE SETTING OF

$x \le y$ if and only if $d(x, y) \le \psi(x) - \psi(y)$.

**Let $\emptyset \neq X_0 \subset X$ be totally ordered, and consider a sequence $x_n \in X_0$ such that**

$\psi(x_n)$ is decreasing to $\alpha := \inf\{\psi(x) : x \in X_0\}$. If $n$

$$\begin{aligned} d(x_{n+m}, x_n) &\le \sum_{i=0}^{m-1} d(x_{n+i+1}, x_{n+i}) \\ &\le \sum_{i=0}^{m-1} \psi(x_{n+i}) - \psi(x_{n+i+1}) \\ &= \psi(x_n) - \psi(x_{n+m}). \end{aligned}$$

$\in \mathbb{N}$ and $m \ge 1$,

Hence $x_n$ is a Cauchy sequence, and admits a limit $x_*$ X, for $X$ is complete. Since $\psi$ can only jump downwards (being lower semicontinuous), we also have $\psi(x_*) = \alpha$. If $x \in X_0$ and $d(x, x_*) > 0$,

then it must be $x \le x_n$ for large $n$. Indeed, $\lim_n \psi(x_n) = \psi(x_*) \le \psi(x)$. We conclude that $x_*$ is an upper bound for $X_0$, and by the Zorn lemma there exists a maximal element $\bar{x}$. On the other hand, $\bar{x} \le f(\bar{x})$, thus the maximality of $\bar{x}$ forces the equality $\bar{x} = f(\bar{x})$.

If we assume that function f is continuous, we get a conclusion that is somewhat more accurate, even if we relax the premise that function f is continuous on $\psi$.

**Theorem** *Let X be a complete metric space, and let f: $X \to X$ be a continuous map. Assume there exists a function $\psi$: $X \to [0, \infty)$ such that*

$d(x, f(x)) \le \psi(x) - \psi(f(x)), \quad \forall x \in X.$

*Then f has a fixed point in X. Moreover, for any $x_0$*

*a fixed point of f.*

PROOF Choose Due to the above condition, the sequence $\psi(f^n(x_0))$ is decreasing, and thus convergent. Reasoning as in the proof of the Caristi theorem, we get that $f^n(x_0)$ admits a limit $\bar{x} \in X$, for $X$ is complete. The continuity of $f$ then entails $f(\bar{x}) = \lim_n f(f^n(x_0)) = \bar{x}$.

**CONCLUSION** –

Our study findings were based on numerous generalisations in the area of "fuzzy metric spaces (FMS)" as well as various "fixed point" outcomes in these spaces. A "fixed point" of a transformation is a point that stays unchanged throughout the transformation. In several domains, "fixed point theory" is primarily employed to explain equilibrium. It is crucial in differential equations, integral equations, partial differential equations, operator equations, and functional equations that occur in several fields such as financial mathematics, stability theory, economics, game theory, best approximation, and dynamic programming. The prominent mathematician Zadeh was familiar with the beneficial concept of "fuzzy sets" (1965). Later, fuzzy logic became the most powerful instrument in a variety of technological domains, including artificial intelligence, computer science, control engineering, medical science, and robotics, among others. Fuzzy set theory is a mathematical breakthrough that allows us to solve a variety of uncertain and real-world situations. Every item in fuzzy set theory has a "degree of membership" between Because it is impossible to compute distance functions with inexact values using traditional metric space theory, Kramosil and Michalek proposed the innovative concept of "fuzzy metric space (FMS)" to solve this issue (1975).

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