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Group/Semi- Group theory and Generalization

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Abstract - This paper is to find original symmetric of group theory and Generalization. The Condition for a group to be given. We give fundamental properties of group and Generalization of semi Group. We also present the is Isomorphism's and Homomorphism of a Group .

Keywords - Group, Semi group, Properties of Group ,Generalization of Semi-Group, concepts of Isomorphism's and Homomorphism of a Group.

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INTRODUCTION

This chapter is a concise mathematical introduction into the algebra of groups. It is build up in the way that definitions are followed by propositions and proofs. The concepts and the terminology introduced here will serve as a basis for the following chapters that deal with group theory in the stricter sense and its application to problems in physics. In Mathematics and abstract algebra group theory studies the algebraic structures known as groups. Group theory and the closely related representation theory have many important applications in physics, chemistry and materials science. Extended and further developed groups give an important approach to a general description of other algebraic structures which have only a binary operation and only a constant. A group is a set of elements which satisfy the four group axioms. The set of elements and group operation $(G,*)$ must satisfy: closure under the operation among elements, every element in the Group G must have an inverse also contained in G, the set must contain an identity element (e) and Associativity must hold for any elements in G. The order of this group of elements is simply the number of elements in the set. Galois introduced into the theory the exceedingly important idea of a [normal] sub-group, and the corresponding division of groups into simple and composite. Moreover by showing that to every equation of finite degree there corresponds a group of finite order on which all the properties of the equation depend, Galois indicated how far reaching the applications of the theory might be and thereby contributed greatly ,if indirectly to its subsequent development. Many additions were made, mainly by French Mathematicians, during the middle part of the [nineteenth] century. There are also special types of groups with elements that act commutatively which have a special name, abelian groups. Certain collections of elements pulled from a group can form a subgroup of G and the order of this group H will divide the order of G as was proven by Lagrange. These simple groups are special to our field of study as they encompass Lie type groups and the sporadic simple groups. These special simple groups are valuable components to our research.

In this paper , we study the properties of extended group to generalize the notion of group /semi group by considering the non –empty subset instead of the identity. In particular ,we get a new algebraic structure which in general is not a group but every group is an group.

REVIEW OF LITERATURE

In this, Section, we give a comprehensive review of the published work that is necessary for this paper. Studies that were guided by these perspectives are also discussed. Respecting the classic –philosophical point of view, Piaget (1970a, 1970b) says that Mathematical abstraction marked by extraordinary detailed and vivid recall of visual images, which gives rise to the knowledge of universals. Piaget (1970a) is searching for the answer to the very difficult and important question about the formation of human knowledge .Referring to the classical view of the problem researchers wonder if all cognitive information has its source in objects, so that the learner is ‘‘ instructed ’’ by the objects or another individual in the world outside him, or on the contrary ,the subject possesses a form produced or growing from within structures which the subject imposes on objects. Further, Piaget (1970a) studies characteristics of cognition: association and assimilation. He critizes the concept of association by claiming that this concept only refers to an external bond between the associated elements. Many additions were made ,mainly by French mathematicians during the middle part of the nineteenth century. The first connected exposition of the theory was given in the third edition of M. Serret,s ‘‘ Cours d,Algebre Superieure,’’ which was published in 1866. This was followed in 1870 by M.Jordan,s ‘‘Traite des substitutions des equations algebriques. ’’ The greater part of M. Jordan,s treatise is devoted to a development of the ideas of

Galois and to their application to the theory of equations. No considerable progress in the theory , as apart from its applications was made till the appearance in 1872 of Herr Sylow,s memoir " Theoremes sur les groups de substitutions" in the fifth volume of the Mathematische Annalen .

PRELIMINARIES

In this section ,we site the fundamental definitions that will be used in the sequel.

Definition 1. A non –empty set G with a binary composition $*$ is called a group , if the following conditions are satisfied:

I. a$*$ b $\in G$ for all a,b $\in$G $(closure\ law)$.

II. a$*(b*c)=(a*b)*c$ for all a,b,c $\in G$ $(Associative\ law)$

III. There exists an element e $\in$ G such that $a*e=e*a=a$ for all a $\in G.(Existence\ of\ identity)$ $(e\ is\ called\ identity\ in\ G)$.

IV. For each a $\in G$, there exists an element $a' \in G$ such that a $*\ a' = a'*a$ =e $(Existence\ of\ inverse)$

$(a'\ is\ called\ inverse\ of\ a\ and\ is\ written\ as\ a' = a^{-1})$

$we\ write$ a group G with respect to binary composition $*$ as $(G,*)$.

$Note$:2. $(I,+),(Q,+),(R,+),(C,+)$ are groups but not $(N,+)$ The set A(S) of all one –to –one mappings of a non-empty set S onto itself is a group with respect to the product of mappings.

Definition 3. Let G be a group and let H be a subset of G . Then H is called a **subgroup** of G if H is itself a group, under the operation induced by G.

Definition 4. Let G be a group ,and let a be any element of G. The set

$$<a>=\{X\in G\ |X=a^n\ for\ some\ n\ \in Z\}$$

Is called the **cyclic Subgroup** generated by a. The group G is called **a cyclic group** if there exists an element a $\in$ G such that G = $<a>$ .In this case a is called **a generator** of G.

Definition 5. Order of an element .Let G be a group , and a $\in$ G. If there exists a least positive integer n such that a^n =e ,then n is called the order of a , written as O(a) =n .If no such positive interger exists, then a is said to be of infinite order.

Definition 6. A group G is called **finite or infinite** according as it contains a finite or infinite number of elements. If a group G contains n elements ,we say that the order of G is n and we write it as O(G) =n.

Definition 7. A group $(G,*)$ is called **abelian or commutative** ,if a $*$ b = b $*$ a for all a,b $\in$ G.

Definition 8. Let G be a group and H $\leq G$.H is a **maximal subgroup** of G if there is no normal subgroup N $\leq G$ such that H $< N < G$.

Definition 9. Let $(G,*)$ be a group. The cardinality of G $(finite\ or\ infinite)$ is defined as the **order of group** and it is denoted by o(G) =$|G|$

Definition 10. A group $(G,*)$ is called abelian or commutative, if a$*b=b*a$ for all a,b $\in G$.

Example 11. The set $s_3=\{i,s_1,s_2,s_3,t_1,t_2\}$ is a non-abelian group of order 6 under the product of two mappings.

Remarks 12. The group s_3 is a rich source of providing counter-example in Group theory.

Example 13: The set S = $\{1,2,3,4\}$ is an abelian group under $\odot_5$ $multiplication\ modulo\ 5$.

The composition table of S with respect to $\odot_5$is as follows:

$\odot_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

From the composition table, we notice that 1 is the identity element of S and the inverse elements of 1,2,3 ,4 are 1,2,3,4 respectively.

Remarks 14: For any positive prime integer p, the set S = $\{1,2,3\ldots,\ p-1\}$ is an abelian group under $\odot_p$ multiplication modulo p.

Definition 15. If a be an element of a group G, then we define a^0 =e and for any positive integer n, a^n is defined as $a^n=a^*a^*a\ldots..a^*$(n times). Also $a^{-n}=(a^{-1})^n$

It is easy to verify that $a^m*a^n=a^{m+n}$ and $(a^m)^n = a^{mn}$, where m and n are any two integers.

PROPERTIES OF GROUPS

We also write the binary composition * as .(. is called the product). The statement that G is a group will now mean that (G,.) is a group.

Theorem 16 : (Cancellation laws) If G is a group and a,b,c ∈ G ,then

(i) a. b = a .c ⇒ b = c (left cancellation law)

(ii) b.a = c. a ⇒ b = c. (right cancellation law)

Theorem 17: If G is a group and a, b ∈ G, then

(i) $(a^{-1})^{-1}$ = a

(ii) $(a.b)^{-1} = b^{-1}.a^{-1}$

Theorem 18 : In a group G, the equations a.x = b and y.a = b have unique solutions for all a,b in G.

Example 19: Let G = $\{a,b,c,d\}$, K = $\{a,b,c\}$ and A = $\{a,b\}$ be three sets with the following tables:

$*_4$	A	B	c	d
a	A	A	a	a
b	A	B	c	D
c	A	C	a	A
d	A	D	c	D

$*_5$	a	B	c
a	**c**	**A**	**b**
b	a	B	c
c	b	C	A

We can easily seen that $(G,*_4;A)$ is a non-abelian group and $(K;*_5;A)$ is an abelian group.

Remark 20 : In example we can see that in general ,A is not a closed subset, since a∈ A but a $*_5$a = c ∉ A.

An group which is not a group will be called proper. Also we will call $(G;*;A)$ a proper group if $|A| > 1$. The following example shows that in general every group is not a group.

Example 21: If G is a group which has a and b as its elements ,such that a,b ∈ G, *then* $(a\times b)^{-1} = a^{-1}\times b^{-1}$

Proof: $(a\times b)\times a^{-1}\times b^{-1}$= I, where I is the identity element of G.

Consider the L.H.S of the above equation, we have,

L.H.S = $(a\times b)\times b^{-1}\times b^{-1}$ ⇒ a × $(b\times b^{-1})\times b^{-1}$

⇒ a × $I \times a^{-1}(by\ associative\ axiom)$

⇒ $(a\times I) \times a^{-1}\ (by\ identity\ axiom)$

⇒ $I(by\ identity\ axiom)$

= R.H.S

Hence ,proved.

GENERALIZATION OF SEMI – GROUP

Definition 22: (Semi-Group) A non-empty set G with a binary composition is called a semi-group, if a.(b.c) = (a.b).c for all a,b,c ϵG

It is clear that every group is a semi-group ,but the converse need not be true.

Semi –group has half of the properties of a group i.e closure and associative only.

Example 23: the set N of natural numbers is a semi-group under multiplication, but N is not a group under multiplication.

Theorem 24: In a semi-group G in which the equations a .x =b and y.a =b are solvable for every pair of elements a,b is a group.

Theorem 25: In a semi-group G is a group if and only if for any a,b ∈ G, the equations a.x = b and y.a =b have solution in G.

Theorem 26 : Suppose a finite set G is closed under an associative product and that both concellation laws hold in G .Then G must be a group.

Remarks 27: The above theorem may not hold in an infinite semi-group.

Example 28 : The set N of natural numbers in an infinite semi-group under addition in which both the cancellation laws hold i.e

a+b = a +c ⇒ b = c , a,b, c ϵN

b+a = c+a ⇒ b = c

but (N,+) is not a group.

Remarks 29: The conclusion of the above theorem may not follow ,if only one of the cancellation laws holds in a finite semi-group G.

Let G be a finite set having at least two elements .Define a composition on G as follows:

a.b = b for all a,b ∈ G ………(i)

for any a,b,c ∈ G, we have

a.(b.c) = a. c = c by (i)

(a.b) .c = b. c = c by (i)

Thus G is a finite semi-group in which left cancellation law holds ,since

a.b = a. c ⇒ b = c using (i)

let a, b be any two distinct elements of G. Then

a.b = b and b.b = b ; using (i)

∴ a.b = b.b but a ≠ b

Thus right cancellation law does not hold in G.

Hence G is not a group.

Theorem 30 : A semi-group (G,.) is a group iff the following conditions are satisfied:

(i) There exists an element e ∈ G such that

a.e = a ∀ a ∈ G

(ii) For each a ∈ G, there exists an element a' ∈ G such that a.a' = e

Remarks 31: The above theorem may be restated as:

A semi-group (G,.) is a group iff

(i) G has a right identity.

(ii) Each element in G has a right inverse in G.

Theorem 32: A semi-group (G,.) is a group iff it satisfies the following conditions:

(i) There exists an element e ∈ G ,such that e.a = a for all a ∈ G. (i.e G has a left identity)

(ii) For each a ∈ G, there exists $a' \in G$ such that a'.a = e (i.e each element in G has a left inverse)

Isomorphism's and Homomorphism of a Group

In this section, we introduce the concept of Isomorphism and Homomorphism finite group through Generalization . An isomorphism is a bijective homomorphism .

Definition 33: A mapping f:$G_1 \to G_1$ is called an isomorphism ,if

(i) F is a homomorphism

(ii) F is one-to-one

i.e f(x) =f(y) ⟹ x=y ; x,y ∈ G

Definition 34: Two groups G_1 and G_2 are called isomorphic, written as $G_1 \approx G_2$, if there is an isomorphism of G_1 *onto* G_2.Equivalently ,two groups G_1 *and* G_2 are isomorphic, if there exists a mapping f:$G_1 \to G_2$ such that

(i) F is a homomorphism ,

(ii) F is one to one

(iii) F is onto

Example 35: Show that $< Q, + >$ cannot be isomorphic to $< Q^*, . >$ where Q^* = Q -{0} and Q = rationals.

Sol: Suppose f is an isomorphism from Q → Q^* . Then as 6 ∈ Q^* f is onto, ∃ α ∈ $< Q, + >$ such that f(α) = 6.

$$\Rightarrow f\left(\frac{\alpha}{6} + \frac{\alpha}{6}\right) = 6$$

$$\text{Or } f\left(\frac{\alpha}{6}\right) f\left(\frac{\alpha}{6}\right) = 6$$

$$\Rightarrow x^2 = 6 \quad where\ x = f\left(\frac{\alpha}{6}\right) \in Q^*$$

But that is a contradiction as there is no rational number x. such that ,x^2 =6

Hence the result follows.

Example 36: For every group G,the identity mapping I_G defined by $I_G : G \to G, I_G(x) = x, \forall x \in G$

Is an isomorphism of G onto itself, because I_G is clearly one –one onto and a,b ∈ G $I_G(ab) = ab$

$= I_G(a) I_G(b)$

Definition 37: Let $(G_1, .)$ and $(G_2, *)$ *be two* groups. A mapping f:$G_1 \to G_2$ is called a homomorphism, if f(a.b) = f(a) ∗ f(b) for all a,b ∈ G_1.

In other words, a homomorphism preserves the compositions in the groups G_1 *and* G_2 .However, If we are not specific about the compositions of the groups G_1 and G_2 ,we say that a mapping

f: $G_1 \to G_2$ is a homomorphism, if

f(ab) =f(a) f(b) for all a,b ∈ G.

Definition 38: A group G_2 is said to be a homomorphic image of a group G_1, if there exists a homomorphism of G_1 onto G_2.

Example 39: If G is a group ,then the mapping f:$G_1 \rightarrow G_2$ defined as f(x) =e for each x $\in G$ is a homomorphism. Let x,y ∈ G so that xy∈ G.

We have f(xy) =e =ee =f(x)f(y)

Hence f is a homomorphism.

CONCLUSIONS

in this paper we generalize the notion of groups to a new algebraic structure as groups and semi-groups. Sub group are defined and some example are given. Suitable Morphisms between such groups which are generalized are considered. These concepts can be further generalized isomorphism and homomorphism and we hope that they may have some applications in the real world.

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