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On Epidemic Dynamics (SI Model)

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Abstract – The present paper provides study of the spread and control of epidemics through a susceptible population.

Key Words: Epidemics, Susceptible, Population, Disease, Infection

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1. INTRODUCTION:

As a matter of fact the study of epidemics has attracted many research workers in field of Biomathematics.

Murray [1993] has described some models concerning population dynamics and for diseases and infections. He has also obtained and tried models for control of diseases and infections.

Bernonli (1760) considered the mathematical description for effect of cowpox and spread of small pox.

Some other workers in the field an Epidemic dynamics and modelling of the infections disease are models can be extremely useful in giving reasoned estimates for the theoretical papers on epidemic models and Kermack and Mc Kendrick (1927, 1932, 1933) Mathamatical models for spread and control of infectious disease are given by Baily (1975) and Hoppensteadt (1975), Wickwire (1977), Anderson (1982).

In this paper, we shall study the spread and control of epidemics through a susceptible population. Here we have focussed on S.I. Model.

2. BASIC CONCEPTS

This spread of a disease depends on the mode of transmission, succeptibility, infections period, resistance any many other factors. That is,

Usually an infectious disease spreads in a population when one or more infectives enter into the population from outburst. Germs of the disease coming from the last outburst of the disease may also manage to survive within the population as, spores which are activated by nature under suitable climatic conditions.

We begin with a population model known as a simple deterministic model. Assume in a given population at time t, S(t) denotes the number of susceptible, I(t) denotes the number of infected persons in the population, and R(t) denotes the number of individuals removed from the population by recovery, death, immunization or other means. Now we will make the following hypothesis:

(i) The disease is transferred by contact between the susceptible and an infected individual.

(ii) The disease is transferred instantaneously.

(iii) All susceptible individuals are equally susceptible and all infected individual are equally infected.

(iv) The population under consideration is fixed Hence, if N is the population size, then S(t) + I(t) + R(t) = N = K (a constant)

SI Model

We consider a simple deterministic epidemic model in which there are no removals from circulation by death, recovery or isolation and each in the population is either infected with the disease or else susceptible to the disease.

(a) Formulation

Let N be size of a population which is considered to be fixed and it is assumed that the susceptible are homogeneously mixing with each other.

Let S_0 be the initial number of susceptibles in the population in which a number of infected individual I_0 have been introduced so that

$$[S(t)]_{t=0} = S_0 \text{ and } [I(t)]_{t=0} = I_0$$

Where S(t) is the number of susceptible and I(t) the number of infectives at any time t.

Then we have

(2.1) $$S(t) + I(t) = S_0 + I_0 = \text{consant} = N(\text{say})$$

Since the population is supposed to be fixed one, therefore because of infection, the number of susceptibles decreases and the number of infected persons increases in number.

If we assume that the rate of decrease of S(t), or the rate increases of I(t) is proportional to the product of the number of susceptibles and the number of infected that

(2.2) $$\frac{dS}{dt} = -\alpha SI$$

and

(2.3) $$\frac{dS}{dt} = -\alpha SI$$

Where α is a positive constant, known as contact rate.

Since these new infective come from the susceptible class, then using equation (2.1), equation (2.2) gives

(2.4) $$\frac{dS}{dt} = -\alpha S(N-S)$$

This equation can be solved separation of variable method.

(b) Solution

Equation (2.4) can be cast in the form

$$\frac{dS}{S(N-s)} = -\alpha dt$$

or $$\left(\frac{1}{N-S} + \frac{1}{S}\right) ds = -\alpha N dt$$

or $$\frac{dS}{N-S} + \frac{dS}{S} = -\alpha N dt$$

Integrating both sides, we get

$$-\log(N-S) + \log S = -\alpha Nt + \log A$$

or $$\log \frac{S}{A(N-S)} = -\alpha Nt$$

or $$\frac{S}{A(N-S)} = e^{-\alpha Nt}$$

or $$A(N-S) = Se^{\alpha Nt}$$

(2.5) or $$S = \frac{An}{A + e^{\alpha Nt}}$$

Initially, when t = 0, S = S_0, so that

$$A = \frac{S_o}{N - S_o}$$

Therefore, (2.5) $$S = \frac{S_o N}{S_o + (N - S_o)e^{\alpha Nt}}$$

Likewise, we obtain

(2.6) $$I = \frac{I_o n e^{\alpha Nt}}{I_0 + (N - I_o)e^{\alpha Nt}}$$

Equation (2.5) provides the number of susceptible at t and equation (2.6) gives the to take t_y of infected persons at the time t.

(c) Discussion

It is informative to look at the limiting behaviour of these solutions.

$$\text{As } t \to \infty, S(t) \to 0 \text{ and } I(t) \to N$$

This result shows that, ultimately all the persons will be infected. By plotting S(t) and I(t) against t, we get graphical representation of equation (2.5) and (2.6) as follows

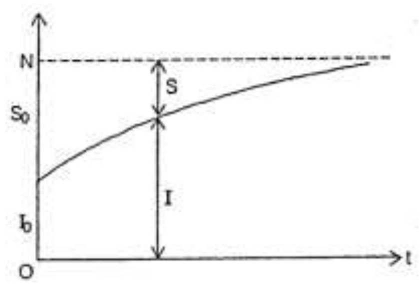

Figure. 1

This suggests that in a high population with low number of infective, I_0, at first the epidemic grows exponentially.

Thus, from this model, we notice that once an epidemic begins, each in the population lastly contracts the disease. This is because infective remain infected forever.

In practice, the public health departments usually record the number of new cases appearing every day or week, i.e., the rate of appearance of new cases, namely, $-\frac{dS}{dt}$

using equation (2.5), we have

$$-\frac{dS}{dt}=\frac{S_0\alpha N^2(N-S_0)e^{\alpha Nt}}{\{S_0+(N-S_0)e^{\alpha Nt}\}^2} \tag{2.7}$$

The rate $\frac{dS}{dt}$ is taken with a negative sign because the number of susceptibles S decreases as the epidemic created. If we draw a curve of the rate of change of the number of susceptibles $\frac{dS}{dt}$ versus t, and the rate of change in the number of infectives, $\frac{dI}{dt}$ versus time t, remembering $\frac{dI}{dt}=-\frac{dS}{dt}$, then we obtain a curve known as the epidemic curve, which is shown in the figure 2

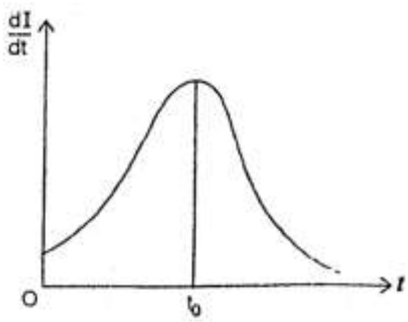

Figure 2

Obviously for given value of N, this curve is symmetrical, therefore, it has extreme value when t = t_0 given by $\frac{d^2S}{dt^2}=0$

Now

$$\frac{d}{dt}\left(\frac{dS}{dt}\right)=-\frac{d}{dt}\left[\frac{S_0\alpha N^2(N-S_0)e^{\alpha Nt}}{\{S_0+(N-S_0)e^{\alpha Nt}\}^2}\right]$$

$$=-\alpha S_0N^2(N-S_0)\frac{d}{dt}\left[\frac{e^{\alpha Nt}}{\{S_0+(N-S_0)e^{\alpha Nt}\}^2}\right]$$

$$=-\alpha S_0N^2(N-S_0)\left[\frac{\{S_0+(N-S_0)e^{\alpha Nt}\}^2\alpha Ne^{\alpha Nt}-2(N-S_0)\alpha Ne^{2\alpha Nt}(S_0+(N-S_0)e^{\alpha Nt}}{\{S_0+(N-S_0)e^{\alpha Nt}\}^4}\right]$$

$$=\frac{-\alpha^2S_0N^3(N-S_0)e^{\alpha Nt}}{\{S_0+(N-S_0)e^{\alpha Nt}\}^3}\left[S_0+(N-S_0)e^{\alpha Nt}-2(N-S_0)e^{\alpha Nt}\right]$$

$$\therefore\ \frac{d^2S}{dt^2}=-\frac{\alpha^2S_0(N-S_0)e^{\alpha Nt}}{\{S_0(N-S_0)e^{\alpha Nt}\}^3}\left[S_0(N-S_0)e^{\alpha Nt}\right]$$

Hence, $\frac{d^2S}{dt^2}=0$ gives

$$S_0=(N-S_0)e^{\alpha Nt_0}$$

or

$$e^{\alpha Nt_0}=\frac{S_0}{N-S_0}$$

$$\Rightarrow t_0=\frac{1}{\alpha N}\log\left(\frac{S_0}{N-S_0}\right) \tag{2.8}$$

Hence, the epidemic curve has a maximum value at $t_0=\frac{1}{\alpha N}\log\left(\frac{S_0}{N-S_0}\right)$ when the number of susceptible as given by equation (2.5)

$$S=\frac{N}{2} \tag{2.9}$$

And from equation (2.7), we have, at t = t_0

$$-\frac{dS}{dt}=\alpha\left(\frac{N}{2}\right)^2 \tag{2.10}$$

Similarly we can discussed SIS model and SIR model.

3. CONTROL OF AN EPIDEMIC

Let an infected person be removed from the scene of disease by quarantine and a susceptible person can be made immune by vaccination. Let vaccination is performed at a rate □. Then our model become.

$$\begin{cases}\frac{dS}{dt}=-\alpha SI-\delta\\ \frac{dI}{dt}=\alpha SI-aI\\ \frac{dR}{dt}=aI\\ \frac{dV}{dt}=\delta\end{cases} \tag{3.1}$$

Where V(t) denotes the number of vaccinated persons at time t. Here, the initial conditions are

(3.2) $$S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = V(0) = 0$$

Obviously

(3.3) $$S(t) + 1(t) + R(t) + V(t) = S_0 + I_0 = N$$

Equation (3.1) can be normalized as

(3.4) $$\begin{cases} \frac{d\bar{S}}{dt} = -\alpha \bar{S}\,\bar{I} - \bar{\delta} \\ \frac{d\bar{I}}{dt} = \alpha \bar{S}\,\bar{I} - a\bar{I} \\ \frac{d\bar{R}}{dt} = a\bar{I} \\ \frac{d\bar{V}}{dt} = \bar{\delta} \end{cases}$$

(3.5) with $\bar{S}(0) = \bar{S}_0, \bar{I}(0) = \bar{I}_0, \bar{R}(0) = \bar{V}(0) = 0$

where

(3.6) $$\bar{S}(t) = \frac{S(t)}{N}, \bar{I}(t) = \frac{I(t)}{N}, \bar{R}(t) = \frac{R(t)}{N}, \bar{V}(t) = \frac{V(t)}{N}$$

also

(3.7) $$\bar{\delta} = \frac{\delta}{N}, \bar{\alpha} = \frac{\alpha}{N}$$

The normalized equation can be obtained from the original equations by putting N = 1. Clearly $\bar{S}, \bar{I}, \bar{R}, \bar{V}$ are the proportions of the populations of the various category.

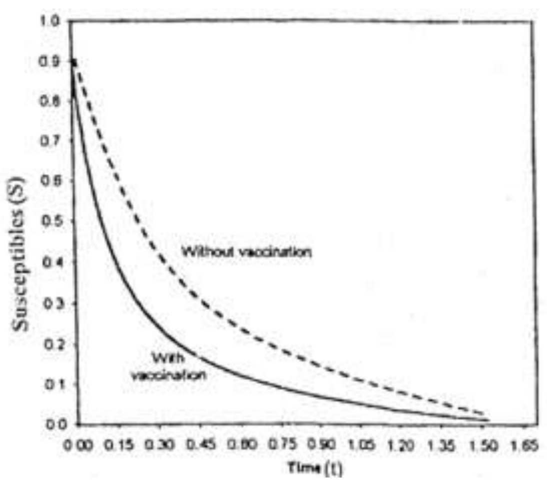

Fig. 3

Here, the control problem arises because vaccination involves costs and we have to minimize the costs for achieving some pre-assigned objects. Also, the cost will depend on the vaccination rate and may be taken as a non-linear function C(α) of the rate α.

Now, we have the following two objectives in controlling an epidemics.

(i) The total proportion $\bar{R}(T) + \bar{I}(T)$ of the population affected by the epidemic over the time (0, T) is less than some prescribed number A.

(ii) The maximum proportion 1m (t) infected at the peak in the interval (0, T) is less than a prescribed number B.

Then, this optimization problem can be stated as follows: "Given a cost function C(α) and the positive constant $\bar{I}_0, \bar{S}_0, \bar{A}, \bar{B}$ and $\bar{T}$, we have to select α(t) such that the system of equations (3.4) – (3.5) gives a solution. Satisfying the conditions $\bar{I}(T) + \bar{R}(T) \le A$.

Max I(T) $\le$ B

[O, T]

C(α) is minimum

This problem can be solved by using the technique of dynamic programming.

Also, from (3.1)

$$\frac{dI}{dS} = \frac{\alpha I(S - \rho)}{-(\alpha IS + \delta)}$$

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